Additive categorification of cluster algebras. Part I

ZMP seminar 15.12.2021 Merlin Christ

Plan

1) Quivers with potential and Ginzburg algebras

2) Examples from triangulated surfaces

3) Ginzburg algebras and BPS states in 4d $\mathcal{N}=2$ QFT

(with many thanks to Murad Alim for help)

Mathematical references

* Derived equivalences from mutations of quivers with potential
  Bernhard Keller and Dong Yang

* Quiver algebras as Fukaya categories
  Ivan Smith

Physical references

* Geometric construction of N=2 gauge theories
  Peter Mayr

* N=2 Quantum Field Theories and Their BPS Quivers.
  Murad Alim, Sergio Cecotti, Clay Cordova, Sam Espahbodi,
  Ashwin Rastogi, Cumrun Vafa

* BPS Quivers and Spectra of Complete N=2 Quantum Field Theories.
  Murad Alim, Sergio Cecotti, Clay Cordova, Sam Espahbodi,
  Ashwin Rastogi, Cumrun Vafa

Recall: A quiver Q consists of vertices and arrows

e.g.

$$\begin{tikzcd} & 2 \arrow[dl] & & 4 \arrow[dr, shift left] & \\ 1 \arrow[rr] & & 3 \arrow[ul] \arrow[ur] \arrow[rr] & & 5 \arrow[ul, shift left] \end{tikzcd}$$

most of the time:

Q has no loops or 2-cycles

k ground field. Path algebra kQ

* k-vector space with basis the paths $a_n \cdots a_1$ in Q

  arrows

  $\mathrm{source}(a_{i+1}) = \mathrm{target}(a_i)$

* product the composition of paths
  (composite $=0$ if paths cannot be composed)

<u>Example</u>

$Q = A_2 = 1 \xrightarrow{a} 2$, $\quad kQ \simeq k^{\oplus 3} = \mathrm{span}\{e_1, e_2, a\}$

lazy path

$a e_1 = a \quad \ldots \qquad \mathrm{unit} = e_1 + e_2$

$a e_2 = 0 = a^2 \quad \ldots$

A potential W on Q is an element of kQ consisting of cycles.

Examples

1) $A_n = \bullet \to \cdots \to \bullet$. No cycles
   n vertices
   $\rightsquigarrow$ all potentials are zero.

2) $C_3 = $
$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd}$$
$W_1 = cba$, $W_2 = bac$

Have equivalence relation on quivers w/ potential, e.g.

- $(C_3, W_1) \sim (C_3, W_2)$ (cyclic equivalence)

- $(C_2 = 1 \underset{b}{\overset{a}{\rightleftarrows}} 2, W = ba) \sim (Q = 1 \quad 2, W = 0)$

  remove "trivial" 2-cycles

Ginzburg algebra of (Q, W)

$\overline{Q}$ is the graded quiver w/

$\sim$ ghost numbers

homological grading convention

* same vertices as Q

* an arrow $a: i \to j$ in degree 0 for each $a: i \to j$ in Q

* an arrow $a^*: j \to i$ in degree 1 for each $a: i \to j$ in Q

* a loop $l_i : i \to i$ in degree 2 for each vertex i of Q

Definition

$\mathcal{G}(Q,W) = (k\overline{Q}, d)$ is the graded path algebra with differential d determined on generators by

* $d(a) = 0$
* $d(a^*) = \partial_a W$ cyclic derivative of W
* $d(l_i) = \sum_a e_i [a, a^*] e_i$ ← lazy path

↑ commutator

Fact: $d^2 = 0$

The cyclic derivative of a cycle c is

$$\partial_a(c) = \sum_{c = uav} vu$$

Example

$$Q = \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd} \qquad W = cba$$

$$d(a^*) = \partial_a(cba) = cb$$

$$d(b^*) = ac \qquad d(c^*) = ba$$

$$\overline{Q} = \begin{tikzcd} & 2 \arrow[loop above, "l_2"] \arrow[dl, bend left, "a^*"] \arrow[dr, bend left, "b"] & \\ 1 \arrow[loop left, "l_1"] \arrow[ur, bend left, "a"] \arrow[rr, "c^*"] & & 3 \arrow[loop right, "l_3"] \arrow[ul, bend left, "b^*"] \arrow[ll, bend left, "c"] \end{tikzcd}$$

$$d(l_1) = -a^*a + cc^*$$

$$d(l_2) = aa^* - b^*b$$

$$d(l_3) = bb^* - c^*c$$

Relation to cluster algebras

Next talk (Jonte)

Additive categorification of cluster algebra of Q via cluster category

$$\mathcal{C}_{(Q,W)} := \mathcal{D}^{\mathrm{perf}}(\mathfrak{g}(Q,W)) / \mathcal{D}^{\mathrm{fin}}(\mathfrak{g}(Q,W))$$

(perfect) derived category of $\mathfrak{g}(Q,W)$-modules

subcategory of finite dimensional modules

Questions

1) Invariance of $\mathcal{C}_{(Q,W)}$ under quiver mutation?
   Dependence on W?

2) How to think about $\mathcal{D}^{\mathrm{perf}}(\mathfrak{g}(Q,W)), \mathcal{D}^{\mathrm{fin}}(\mathfrak{g}(Q,W))$?

Recall (see talk 1): mutations of quivers

i vertex of Q $\qquad$ $\mu_i(Q)$ mutatated quiver w/

* same vertices as Q

(1) For every subquiver $k \xrightarrow{a} i \xrightarrow{b} \ell$ add a new arrow $[ba] : k \to \ell$

(2) Reverse all arrows with source or target i

(3) Delete maximal set of pairwise edge-disjoint 2-cycles

Quiver mutation extends to quivers with potential

<u>Example</u>

$$Q = \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \end{tikzcd} \qquad \mu_2(Q) = \begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "{[ba]}"'] & & 3 \arrow[ul, "b"'] \end{tikzcd}$$

$$W = 0 \qquad \mu_2(W) = [ba]ab$$

Main difference: In step (3) only "trivial" 2-cycles are removed

↳ depends on potential

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd} \qquad W = ba$$

Choose W so that no 2-cycles appear upon mutation.

Often: unique such choice possible (up to equivalence)

Theorem (Keller–Yang)

There exists an equivalence of categories

$$\mu_i : \mathcal{D}^{\mathrm{perf}}(\mathcal{G}(Q,W)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}(\mathcal{G}(\mu_i(Q),\mu_i(W)))$$

restricting to an equivalence

$$\mathcal{D}^{\mathrm{fin}}(\mathcal{G}(Q,W)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{fin}}(\mathcal{G}(\mu_i(Q),\mu_i(W)))$$

Thus have

$$\mathcal{C}_{(Q,W)} \xrightarrow{\sim} \mathcal{C}_{(\mu_i(Q),\mu_i(W))}$$

2) For each vertex i of Q have:

* indecomposable projective $\mathcal{G}(Q,W)$-module

$$P_i = e_i\,\mathcal{G}(Q,W) \quad \leftarrow \text{sums of paths starting at } i$$

Have $\bigoplus_{j \text{ vertex}} P_j = \mathcal{G}(Q,W)$ and the P_j's generate $\mathcal{D}^{\mathrm{perf}}(\mathcal{G}(Q,W))$

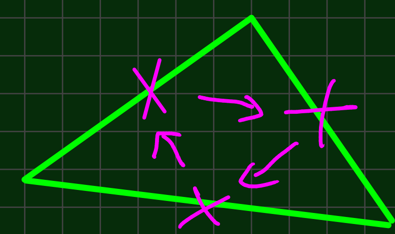
* indecomposable simple $\mathcal{G}(Q,W)$-module $S_i = ke_i$ ← 1-dimensional

The S_j's generate $\mathcal{D}^{\mathrm{fin}}(\mathcal{G}(Q,W))$.

Examples from triangulated surfaces

- S oriented compact surface w/ boundary (possibly empty)
- $M \subset S$ finite set of marked points
- Triangulation τ of S: decomposition into triangles with vertices (= corners) the marked points

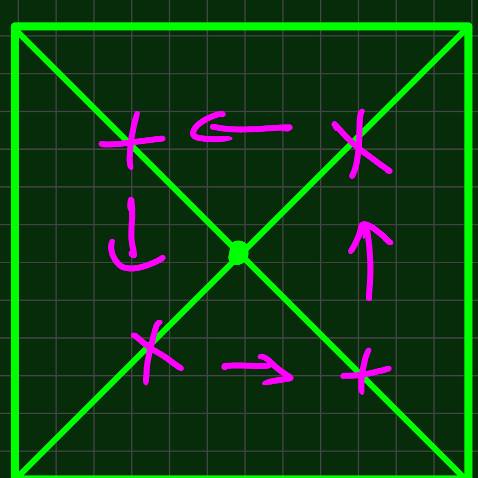
Associated quiver w/ potential (Q_τ, W_τ)



* vertices of Q_τ = interior edges of τ
* arrows obtained from inscribing a clockwise 3-cycle into each triangle
* W_τ consists of clockwise 3-cycles and counterclockwise n-cycles

Examples

4-gon w/ one interior marked point

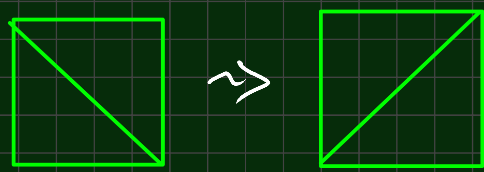


$$Q_\tau = \begin{tikzcd} 2 \arrow[d, "b"'] & 1 \arrow[l, "a"'] \\ 3 \arrow[r, "c"] & 4 \arrow[u, "d"'] \end{tikzcd}$$

mutation equivalent to D_4

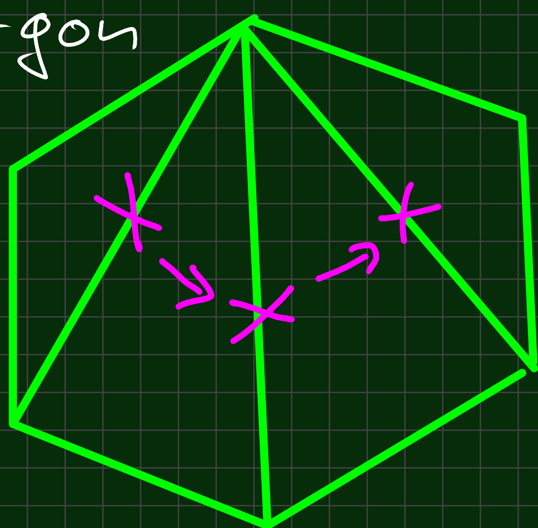
$$W_\tau = dcba$$

flip triangulation T at an edge i $\quad\longleftrightarrow\quad$ mutate (Q_T, W_T) at vertex i

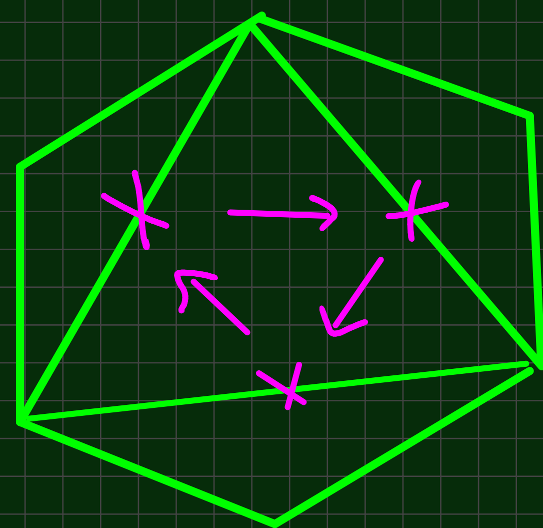
locally 

<u>Example</u>

6-gon



flip $\rightsquigarrow$



$$Q_T = \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \end{tikzcd} \quad \xrightarrow{\text{mutate}} \quad Q_T = \begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 \arrow[rr, "{[ba]}"'] & & 3 \arrow[ul, "b"'] \end{tikzcd}$$

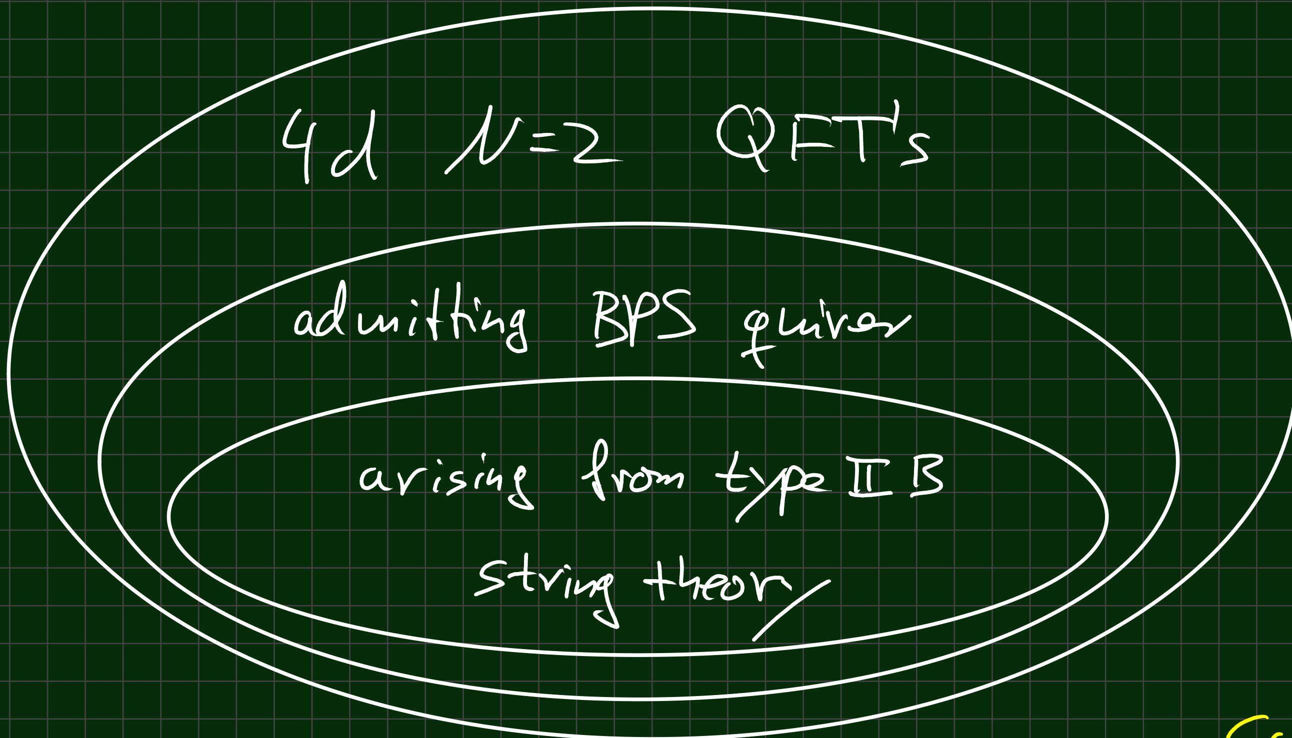
$W_T = 0$ $\qquad\qquad$ $W_T = [ba]ab$ $\quad$ 3-cycle

Can connect any two triangulations of S by flips

$\rightsquigarrow$ well-defined cluster algebra and cluster category of surface!

Relation to 4d $\mathcal{N}=2$ QFT's from string theory

BPS states in supersymmetric QFT: minimize central charge



Calabi-Yau threefold

10d IIB string theory on $M_4 \times Y$

Minkowski space

decouple gravity $\rightsquigarrow$ 4d $\mathcal{N}=2$ QFT w/

BPS states = special Lagrangians in Y

$\equiv$ stable w.r.t. stability condition

D3 branes lie in topological sector $\leftrightsquigarrow$ A-branes

Have fibration $\pi: Y \to \tilde{S}$

Riemann surface / marked surface S + bad singularities

Simplest case: fibres of Y are of type A_1

Then:

* BPS-quiver = quiver arising from triangulation of S

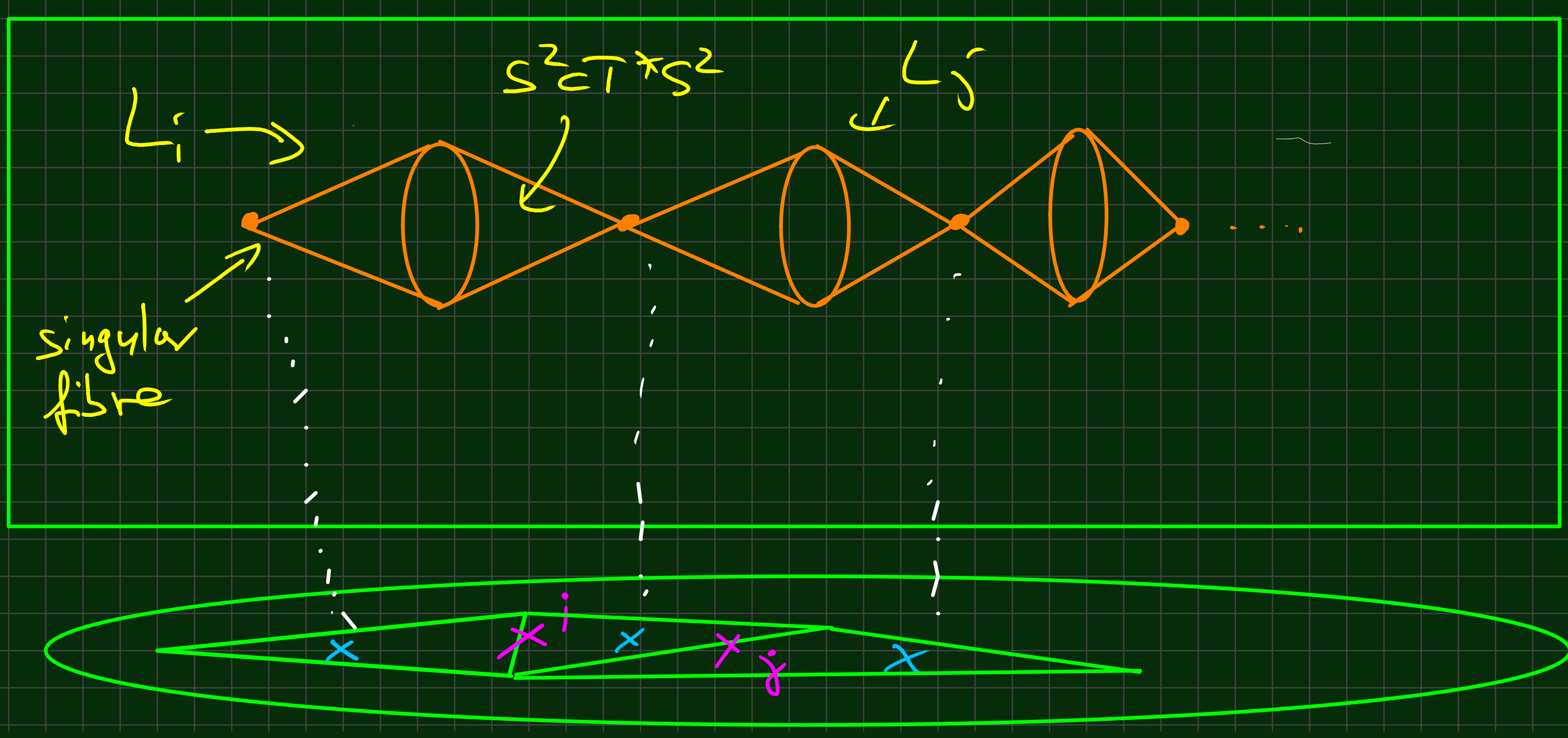
* generic fibre of π: affine conic $T^*S^2 \simeq A_1$-Milnor fibre.
* singular fibres the A_1-singularity $\{ab - c^2 = 0\} \subset \mathbb{C}^3$

<u>Theorem (Smith)</u> ← mathematical theorem

Derived category of A-branes on Y (Fukaya category) $\simeq \mathcal{D}^{fin}(\mathcal{G}(Q_\tau, W_\tau))$

<u>Applications</u>

1) Interpret simple module S_i as spherical brane L_i



$L_i = \Sigma S^2 \simeq S^3$
suspension of 2-sphere

2) Determine BPS spectrum by mutation

3) Stability conditions: Wall crossing models quiver mutation

Thank you

for listening!