Singularities of scattering amplitudes from cluster algebras & their generalizations
– Part 2 –

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ZMP Seminar 2021

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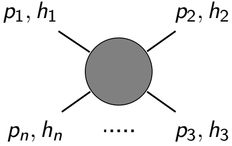
based on [JHEP 08 (2020) 005, JHEP 10 (2021) 007] (with Georgios Papathanasiou)

Outline

Scattering amplitudes in $\mathcal{N} = 4$ pSYM

Amplitudes

Scattering amplitudes are fundamental **observables**, quantities which can be measured, that give the probability for certain outcomes in the interaction of fundamental particles.



$\mathcal{N} = 4$ **super Yang-Mills theory**

We consider the simplex interacting gauge theory with color group $SU(N_c)$, coupling constant g, and maximal amount of $\mathcal{N} = 4$ supersymmetries. We also take the planar limit, where $N_c \to \infty$ while gN_c^2 stays constant.

$\mathcal{N} = 4$ pSYM loop amplitudes & cluster algebras

The L-loop amplitude is given by weight-$2L$ multiple polylogarithms.

Key observation

The letters φ_{α_i} of $\mathcal{N} = 4$ pSYM loop amplitudes for $n = 6, 7$ are cluster $\mathcal{A}$-variables of the *cluster algebra* associated to $\mathrm{Gr}(4, n)$.
[Golden, Goncharov, Spradlin, Vergu, Volovich '13]

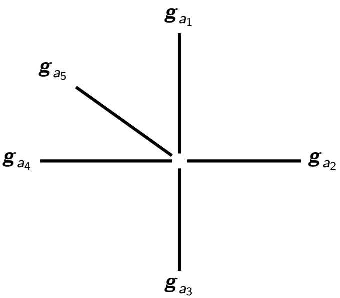
Crucial input for cluster bootstrap:

1. "Guess" the collection of letters, the *alphabet*, of the amplitude
2. Construct the space of all weight $2L$ symbols
3. Fix amplitude with consistency conditions and physical constraints
4. "Integrate" the symbol to obtain the amplitude as a function

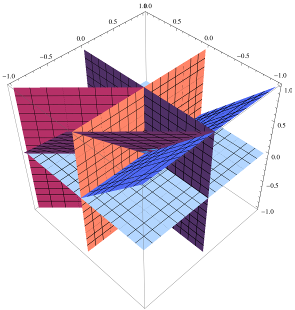
see eg. [Dixon, Drummond, Henn '11], [Drummond, Papathanasiou, Spradlin '14] [Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdoğan, von Hippel, McLeod, Papathanasiou '20]

Cluster fan

A_2



A_3



Definition

To each cluster algebra we can associate a *cluster fan*, with associations

$$\text{clusters} \longleftrightarrow \textit{cones}\,, \quad \text{variables} \longleftrightarrow \textit{rays}$$

see also [Fomin, Zelevinsky '01]

Finite selection rule for infinite cluster algebras

Problem

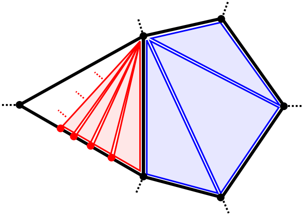
Cluster algebra of $\mathrm{Gr}(4,8)$ predicts infinitely many singularities for $\mathcal{A}_8$.

Solution

The positive tropical kinematic space $\widetilde{\mathrm{Tr}}_+(4,8)$ comes to the rescue via a selection rule for the cluster algebra!

Revisiting the triangulation of $F_{k,n}$:

- The cluster fan triangulates the tropical fan.
- This triangulation may contain redundant rays, infinitely triangulating an already triangular cone.



Selection rule

Stop mutating the cluster algebra, whenever you encounter a cluster containing redundant rays.

Outline

1. Scattering amplitudes, multiple polylogarithms and symbols
2. Cluster algebras
3. Six- and seven-particle amplitudes
4. Tropical cluster algebras
5. Eight-particle & nine-particle amplitudes
6. Infinite mutation sequences
7. Conclusions & Outlook

Challenge: it's an irrational world!

Cluster variables

$\mathcal{A}$-variables of a cluster algebra are always rational functions of the initial variables (eg. Plücker variables $\langle ijkl \rangle$) due to the mutation relation

$$a_j \to a_j' = \frac{\prod_{i \to j} a_i + \prod_{j \to i} a_i}{a_j} \,.$$

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Square-root letters

Cluster algebras hence cannot describe square-root letters, such as $\sqrt{\Delta_{ijkl}}$ whereas

$$\Delta_{ijkl} = (f_{ij} f_{kl} - f_{ik} f_{jl} - f_{il} f_{jk})^2 - 4 f_{ij} f_{kl} f_{kl} f_{il} \, ,$$
$$f_{ij} = \langle i i+1 j j+1 \rangle \, ,$$

which is known to appear in the symbol of eight-particle amplitudes!

(see eg. [He, Li, Zhang '19'20; Li, Zhang '21])

Square-root letters from infinite mutation sequences

For the infinite mutation sequence of the affine $A_1^{(1)}$ cluster algebra

$$a_1 \rightrightarrows a_2 \quad \xrightarrow{\mu_1} \quad a_3 \leftleftarrows a_2 \quad \xrightarrow{\mu_2} \quad a_3 \rightrightarrows a_4 \quad \xrightarrow{\mu_2} \quad \cdots$$

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we get a recursion relation for the variables a_i, which can be solved for

$$\lim \frac{a_i}{a_{i-1}} = \frac{a_2}{2a_1}\left(1 + x_1 + x_1x_2 + \sqrt{(1 + x_1 + x_1x_2)^2 - 4x_1x_2}\right) ,$$

where $x_1 = 1/a_2^2, x_2 = a_1^2$. [Canakci, Schiffler '16]

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Square-root letters

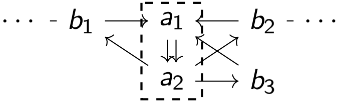
Including (the limit of) infinite mutation sequences, we obtain generalised cluster variables that correspond to the square-root letters of $\mathcal{A}_8$.

(see eg. [NH, Papathanasiou '19],[Arkani-Hamed, Lam, Spradlin '19],[Drummond, Foster, Gürdoğan, Kalousios '19])

Algebraic alphabet of eight-particle amplitudes

Origin clusters

In the cluster algebra of $\mathrm{Gr}(4,8)$ truncated by $\widetilde{\mathrm{pTr}}_{+}(4,8)$, we find 3200 $A_1^{(1)}$ *origin clusters* of the following type



Of these origin clusters, 32 are distinct with respect to the limit for each of the 2 limit rays.

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$$\begin{tikzcd}
\cdots \arrow[r, dashed, no head] & b_1 \arrow[r] & a_1 \arrow[d, shift left] \arrow[d, shift right] & b_2 \arrow[l] \arrow[r, dashed, no head] & \cdots \\
 & & a_2 \arrow[ul] \arrow[r] \arrow[ur] & b_3 \arrow[ul] &
\end{tikzcd}$$

Of these origin clusters, 32 are distinct with respect to the limit for each of the 2 limit rays.

Algebraic 8-particle alphabet

The 64 origin clusters give rise to 128 algebraic letters which can be reduced to the **18 multiplicatively independent algebraic letters** previously found.

[NH, Papathanasiou '19], [Drummond, Foster, Gürdoğan, Kalousios '19], [Zhang, Li, He '19]

The fine print: dependence on frozen variables

We mutate the A_2 cluster algebra with frozen variables at a_2 to get

$$\begin{tikzcd} \boxed{f_1} \arrow[dr] & \boxed{f_2} \\ a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] \end{tikzcd}$$

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a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] & {} \arrow[r, "\mu_2"] & a_1 \arrow[u] \arrow[ur] & a_2' \arrow[l] \arrow[ul]
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$$a_2' = \frac{f_1 a_1 + f_2}{a_2}$$

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Had we chosen different frozen variables, the new variable a_2'' would be

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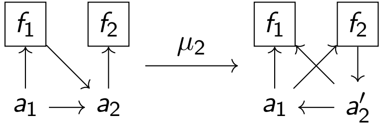

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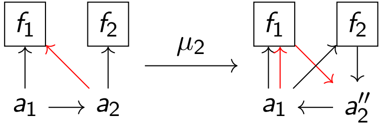

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Challenge

Due to different frozen variables, we have to compute the limit of the mutation sequence separately for each embedding of $A_1^{(1)}$ in $\mathrm{Gr}(4, n)$.

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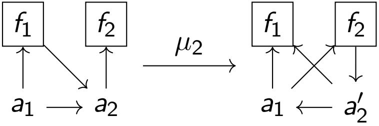

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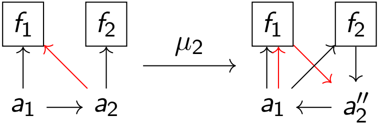

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Due to different frozen variables, we have to compute the limit of the mutation sequence separately for each embedding of $A_1^{(1)}$ in $\mathrm{Gr}(4, n)$.

$\Rightarrow$ Using so-called **coefficients** instead of frozen variables solves this!

Cluster algebras with coefficients

Coefficients

For a cluster with variables $a_1, \ldots, a_n$ and frozen variables $f_{n+1}, \ldots, f_{n+m}$, and adjacency matrix b_{ij}, the *coefficients* $y_1, \ldots, y_n$ are

$$y_i = \prod_{j=n+1}^{n+m} f_j^{b_{ji}} \,.$$

Mutation (coefficients)

When mutating at node j, coefficients mutate as

$$y_j \to y_j^{-1}\,, \quad y_i \to y_i y_j^{\max(0,b_{ji})} \left(1 \hat{\oplus} y_j\right)^{-b_{ji}} \text{ if } i \neq j\,,$$

where the *cluster-tropical addition* is defined on the frozen variables as

$$\prod_i (f_i)^{b_i} \hat{\oplus} \prod_j (f_j)^{c_j} = \prod_i (f_i)^{\min(b_i, c_i)} \,.$$

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Mutation ($\mathcal{A}$-variables)

The mutation rule of the $\mathcal{A}$-variables under a mutation at node j is changed to

$$a_j = \frac{y_j \prod_{i \to j} a_i^{b_{ij}} + \prod_{j \to i} a_i^{b_{ij}}}{\left(1 \hat{\oplus} y_j\right) a_j} .$$

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Relation to $\mathcal{X}$-variables

The coefficients are related to the $\mathcal{X}$-variables of the cluster by

$$x_i = \prod_{j=1}^{n} a_j^{b_{ji}} \cdot y_i \,.$$

Cluster algebras with coefficients – examples

$$\begin{tikzcd} \boxed{f_1} \arrow[dr] & \boxed{f_2} \\ a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] \end{tikzcd} \qquad\qquad \begin{tikzcd} \boxed{f_1} & \boxed{f_2} \\ a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] \arrow[ul, red] \end{tikzcd}$$

The coefficients of the initial cluster are given by

$$y_1 = f_1^{-1}\,, \quad y_2 = f_1 f_2^{-1}\,, \qquad\qquad y_1 = f_1^{-1}\,, \quad y_2 = \color{red}{f_1^{-1} f_2^{-1}}\color{black}\,,$$

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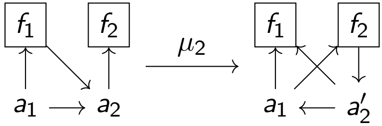

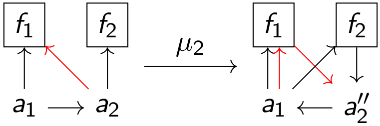

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After mutation, we get the modified coefficients as

$$y_1' = y_1(1 \hat{\oplus} y_2) = f_1^{-1} f_2^{-1}\,, \qquad\qquad y_1' = y_1(1 \hat{\oplus} y_2) = f_1^{-1}\,,$$
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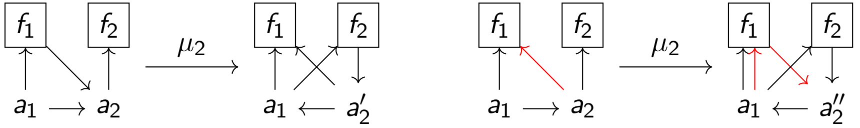

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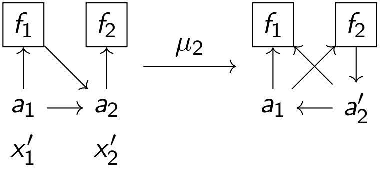
$$y_2' = y_2^{-1} = f_1^{-1} f_2\,, \qquad\qquad y_2' = y_2^{-1} = f_1 f_2\,.$$

and the new $\mathcal{A}$-variable as

$$a_2' = \frac{y_2 a_1 + 1}{a_2 \left(1 \hat{\oplus} y_2\right)} = \frac{f_1 a_1 + f_2}{a_2}\,, \qquad\qquad a_2' = \frac{y_2 a_1 + 1}{a_2 \left(1 \hat{\oplus} y_2\right)} = \frac{a_1 + f_1 f_2}{a_2}\,.$$

Interlude: separation principle and cluster rays



The $\mathcal{X}$-variables of the initial cluster are given by

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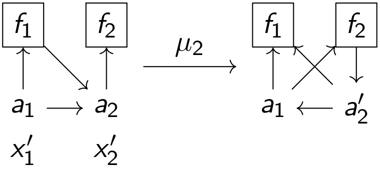

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Using this, we can rewrite the mutated variable a_2' as

$$a_2' = \frac{y_2 a_1 + 1}{a_2 \left(1 \hat{\oplus} y_2\right)} = \frac{1}{a_2} \cdot \frac{1 + x_2}{1 \hat{\oplus} y_2} \,.$$

Interlude: separation principle and cluster rays

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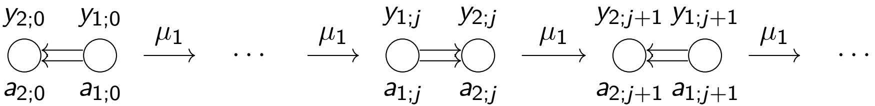
Separation principle

In general, for (well-behaved) cluster algebras, we can write any $\mathcal{A}$-variable a as

$$a = \prod_{i=1}^{r} a_i^{g_i} \cdot \frac{F(x_1, \ldots, x_r)}{F_{\mathbb{T}}(y_1, \ldots, y_r)} \,.$$

The collection of exponents $g = (g_1, \ldots, g_r)$ – the g-vector – is the **cluster ray** associated to a!

Affine rank-2 cluster algebra $A_1^{(1)}$ with general coefficients



Recursion relation

The variables and coefficients along the sequence are related by

$$a_{2;j+1} = \frac{a_{2;j}^2}{a_{1;j}} \frac{1 + x_{1;j}}{1 \hat{\oplus} y_{1;j}} , \quad a_{1;j+1} = a_{2;j} ,$$

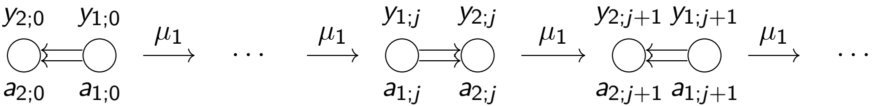
$$y_{1;j+1} = \frac{y_{2;j}\, y_{1;j}^2}{\left(1 \hat{\oplus} y_{1;j}\right)^2} , \quad y_{2;j+1} = \left(y_{1;j}\right)^{-1} .$$

We also introduce the auxiliary sequences

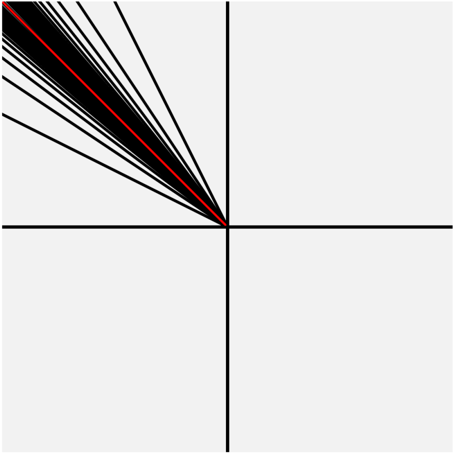
$$\gamma_j = 1 \hat{\oplus} y_{1;j} \hat{\oplus} y_{1;j} \left(y_{1;j-1}\right)^{-1} , \quad \beta_j = \frac{a_{2;j}}{a_{1;j}} .$$

see [NH, Papathanasiou '20]

Affine rank-2 cluster algebra $A_1^{(1)}$ with general coefficients



Fan of $A_1^{(1)}$



Black lines: rays of $A_1^{(1)}$

Red line: limit of rays associated to $a_{1;j}$ and $a_{2;j}$

see eg. [NH, Papathanasiou '20], [Canakci, Schiffler '16], [Reading '17], [Fordy, Marsh '09]

Invariants and solution of $A_1^{(1)}$ mutation sequence

Invariants

Along the sequence, the following two quantities are invariant

$$K_1 = \left(\gamma_0 \gamma_j^{-1} \beta_0^{-1} \beta_j\right) \left[1 + x_{1;j} + x_{1;j}(x_{1;j-1})^{-1}\right] = 1 + x_{1;0} + x_{1;0} x_{2;0}\,,$$

$$K_2 = \left(\gamma_0 \gamma_j^{-1} \beta_0^{-1} \beta_j\right)^2 \left[x_{1;j}(x_{1;j-1})^{-1}\right] = x_{1;0} x_{2;0}\,,$$

with respect to the two $\mathcal{X}$-variables x_1, x_2 of $A_1^{(1)}$ within the *origin cluster*.

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$$K_2 = \left(\gamma_0 \gamma_j^{-1} \beta_0^{-1} \beta_j\right)^2 \left[x_{1;j}(x_{1;j-1})^{-1}\right] = x_{1;0} x_{2;0} \,,$$

with respect to the two $\mathcal{X}$-variables x_1, x_2 of $A_1^{(1)}$ within the *origin cluster*.

Solution

The recursion relation is solved for $a_{1;j}$ by

$$a_{1;j} = (\gamma_0 \cdots \gamma_{j-1}) \left[C_+ (\beta_+)^j + C_- (\beta_-)^j\right] \,,$$
$$C_\pm = a_{1;0} \frac{\pm 2 \mp K_1 + \sqrt{K_1^2 - 4K_2}}{2\sqrt{K_1^2 - 4K_2}} \,, \; \beta_\pm = \frac{a_{2;0}}{a_{1;0}} \frac{K_1 \pm \sqrt{K_1^2 - 4K_2}}{2\gamma_0} \,.$$

Infinite mutation sequences & square-root letters

Algebraic letters

Adopting the proposal of [Drummond, Foster, Gürdoğan, Kalousios '19], we associate the two algebraic letters

$$\phi_1 = \frac{C_+}{C_-} = \frac{K_1 - 2 + \sqrt{K_1^2 - 4K_2}}{K_1 - 2 - \sqrt{K_1^2 - 4K_2}}\,,$$

$$\phi_2 = \frac{\tilde{C}_+}{\tilde{C}_-} = \frac{K_1 - 2K_2 + \sqrt{K_1^2 - 4K_2}}{K_1 - 2K_2 - \sqrt{K_1^2 - 4K_2}}\,,$$

to each $A_1^{(1)}$ cluster appearing in $\mathrm{Gr}(4, n)$.

Infinite mutation sequences & square-root letters

Algebraic letters

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In particular, this general formulation can be applied to the scattering of any number n of particles!

Application: nine-particle alphabet

Rational nine-particle alphabet

Applying the finite selection rule obtained from the tropical configuration space $\widetilde{\mathrm{pTr}}_+(4,9)$, we obtain **3078** cluster variables in one-to-one correspondence to the tropical rays.
These letters include the 531 rational letters previously known in the literature [He, Li, Zhang '20].

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Algebraic nine-particle alphabet

The solutions of all $A_1^{(1)}$ mutation sequences in the (truncated) cluster algebra of $\mathrm{Gr}(4,9)$, we find **2349** square-root letters associated to 324 tropical rays.
These contain the 99 algebraic letters previously known in the literature [He, Li, Zhang '20].

see [NH, Papathanasiou '20]

Nine-particle alphabet: new challenges

Rational alphabet

1. Much more complicated rational letters, with some being polynomials with tens of thousands of terms.
2. Bootstrap becomes infeasible, as the linear spaces the approach is based on are of size $(\#\text{ letters})^{2L}$.

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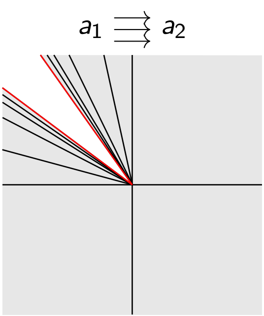
Algebraic alphabet

1. Infinite sequences of the type $A_1^{(1)}$ yield 324 rays in addition to the rational rays.
2. 27 rays of $\widetilde{\mathrm{pTr}}_+(4,9)$ are not accessible from the cluster fan.
3. This is known in the mathematics literature due to eg. the fan of the cluster algebra to the right, which has higher-dimensional holes.



Outline

Conclusions & Outlook

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- The relation to tropical Grassmannians provides a finite selection rule
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- These ideas are successfully applied to obtain the alphabet for $n = 8, 9$, which is in agreement to the literature!

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Some questions remain:

- Which geometry actually describes loop amplitudes of $\mathcal{N} = 4$ pSYM?
- How can bootstrapping the amplitudes made feasible?
- Is there a way to access the 27 still missing rays?