Singularities of scattering amplitudes from cluster algebras & their generalizations

Niklas Henke

ZMP Seminar 2021

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based on [JHEP 08 (2020) 005, JHEP 10 (2021) 007] (with Georgios Papathanasiou

Outline

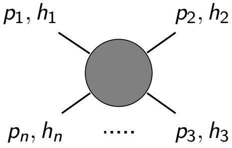
1. Scattering amplitudes, multiple polylogarithms and symbols
2. Cluster algebras
3. Six- and seven-particle amplitudes
4. Tropical cluster algebras
5. Eight-particle & nine-particle amplitudes
6. Infinite mutation sequences
7. Conclusions & Outlook
8. Bonus material

Outline

Scattering amplitudes

Amplitudes

Scattering amplitudes are fundamental **observables**, quantities which can be measured, that give the probability for certain outcomes in the interaction of fundamental particles.



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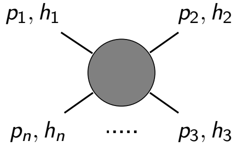
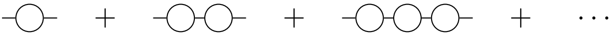


Computing amplitudes

Conventionally, amplitudes are computed as a perturbative expansion:



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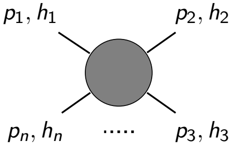

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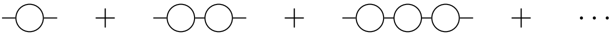

But: The leading term of eg. $g + g \to 8g$ requires millions of diagrams!

Modern amplitudes methods

Computing amplitudes (efficiently)

Scattering amplitudes have intricate mathematical structures, that can be exploited to skip the perturbative expansion entirely!

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Amplitudes & maths

In this way, amplitudes bridge from experimental and theoretical physics to pure mathematics, by eg. asking the questions:

- What numbers / functions can appear?
- What are the algebraic / analytic properties of these functions?
- Can general mathematical statements about the functions in turn be used to compute amplitudes?
- Do (geometric) structures exist, that capture the amplitude without relying on a perturbative expansion?

Planar $\mathcal{N} = 4$ super Yang-Mills

$\mathcal{N} = 4$ super Yang-Mills theory

We consider the gauge theory with color group $\mathrm{SU}(N_c)$, coupling constant g, and maximal amount of $\mathcal{N} = 4$ supersymmetries – the *simplest* interacting gauge theory – to answer these questions.

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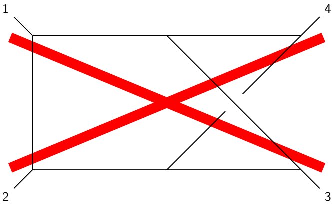
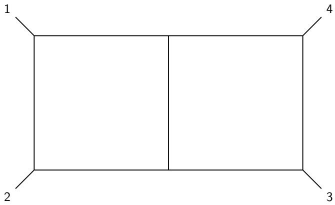
Planar limit

Taking the number of colors $N_c \to \infty$ while keeping the *'t Hooft coupling* $\lambda = g^2 N_c$ constant, only planar Feynman graphs contribute to the amplitude – which makes our job much less complicated!





Amplitudes of $\mathcal{N} = 4$ pSYM

For $\mathcal{N} = 4$ pSYM, the *color-ordered* amplitude is given by

$$\mathcal{A}_n \propto \sum_{L=0}^{\infty} \sum_{k=0}^{n-4} (g^2 N_c)^L \mathcal{A}_{n,k}^{(L)}(x_1, \ldots, x_{3n-15})$$

where

- n – number of particles
- L – number of loops / order in pertrubative expansion
- k – helicity configuration (all but $k+2$ particles with positive helicity)
- x_i, $i = 1, \ldots, x_{3n-15}$ – variables of the space of kinematics

Kinematics of $\mathcal{N} = 4$ amplitudes & Grassmannians

Momentum twistors

The kinematics can be parameterised in terms of momentum twistors $Z_i \in \mathbb{CP}^3$, $i = 1, \ldots, n$. These transform in the fundamental representation of $\mathrm{SL}(4, \mathbb{C})$, which corresponds to dual conformal symmetry. The basic $\mathrm{SL}(4, \mathbb{C})$-invariants are the *Plücker variables*

$$\langle ijkl \rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \det\left(Z_i Z_j Z_k Z_l\right)$$

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Kinematic space

The kinematic space is the configuration space of n points in $\mathbb{CP}^3$, or

$$\mathrm{Conf}(\mathbb{P}^3) \simeq \mathrm{Gr}(4,n)/(\mathbb{C}^*)^{n-1} =: \widetilde{\mathrm{Gr}}(4,n)\,,$$

with the Grassmannian $\mathrm{Gr}(k,n)$ being the space of k-dimensional planes in a n-dimensional vector space.

Functions of $\mathcal{N} = 4$ pSYM loop amplitudes

Observation

All known L-loop amplitudes are *multiple polylogarithms* of weight $2L$.

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Definition (differential)

A multiple polylogarithm (MPL) F_w of weight w is recursively defined as

$$dF_w = \sum_{\varphi_\alpha} F^\alpha_{w-1} \cdot d\log\varphi_\alpha \,,$$

for a collection of rational functions φ_α and MPLs F^α_{w-1} of weight $w-1$.

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Definition (integral)

Multiple polylogarithms can equivalently be defined as iterated integrals. A weight w MPL I_w is given by

$$I_w \left(a_0; a_1, \ldots, a_w; a_{w+1}\right) = \int_{a_0}^{a_{w+1}} \frac{dt}{t - a_w} I_{w-1} \left(a_0; a_1, \ldots, a_{w-1}; t\right) \, ,$$

Symbols and alphabets

Definition

The *symbol map* on polylogarithms is defined via

$$\mathcal{S}\left[F_w\right] = \sum_{\varphi_\alpha} \mathcal{S}\left[F^\alpha_{w-1}\right] \otimes \log \varphi_\alpha$$

with the **letters** φ_α encoding the branch-cut structure of the function.

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Example

The dilogarithm is of weight 2 and is given by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} dt$$

The symbol of the dilogarithm is

$$\mathcal{S}\left[\mathrm{Li}_2\right] = -\log(1-x) \otimes \log(x)\,.$$

Hopf algebra

Definition

A *coalgebra* is a K-vector space V, which is equipped with a *coproduct* $\Delta : V \to V \otimes V$ and *counit* $\epsilon : V \to K$ satisfying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \tag{1}$$

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Definition

A *Hopf algebra* is a K-vector space $\mathcal{H}$ which is both an algebra and a coalgebra, with certain compatibility conditions such as

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)\,,$$

and is further equipped with an *antipode map* $S : \mathcal{H} \to \mathcal{H}$ satisfying

$$S(a \cdot b) = S(b) \cdot S(a)\,, \quad \mu(\mathrm{id} \otimes S) \circ \Delta = \mu(S \otimes \mathrm{id}) \circ \Delta = 0\,.$$

Hopf algebra of MPLs

Claim

The space of all MPLs $\mathcal{H}$ forms a Hopf algebra graded by weight

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \,,$$

where $\mathcal{H}_0 = \mathbb{Q}$ and $\mathcal{H}_n$ consists of MPLs of weight n.

Hopf algebra of MPLs (shuffle product)

Shuffle product formula

The product of two MPLs of weight n and m is given by

$$I_n(a; z_1, \dots, z_n; b) \cdot I_m(a; z_{n+1}, \dots, z_{n+m}; b) = \sum_{\sigma \in \Sigma_{m,n}} I_{n+m}(a; z_{\sigma(1)}, \dots, z_{\sigma(n+m)}; b)$$

where $\Sigma_{m,n}$ are all permutations that separately respect the ordering of $1, \dots, n$ and $n+1, \dots, n+m$.

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where $\Sigma_{m,n}$ are all permutations that separately respect the ordering of $1, \dots, n$ and $n+1, \dots, n+m$.

Example

The product of a generic weight-2 with a weight-1 MPL is given by

$$I_2(a; z_1, z_2; b) \cdot I_1(a; z_3; b) = I_3(a; z_3, z_1, z_2; b) + I_3(a; z_1, z_3, z_2; b) + I_3(a; z_1, z_2, z_3; b)$$

Hopf algebra of MPLs (coproduct)

Coproduct

The coproduct on MPLs is defined as

$$\Delta\left(I\left(a_0; a_1, \ldots, a_n; a_{n+1}\right)\right) =$$

$$\sum_{0=i_1<\cdots<i_{k+1}=n} I\left(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}\right) \otimes$$

$$\prod_{p=1}^{k} I\left(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{p+1}\right) .$$

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Example

The coproduct of the Dilogarithm $\mathrm{Li}_2(x)$ is given by

$$\Delta\left(\mathrm{Li}_2(x)\right) = 1 \otimes \mathrm{Li}_2(x) + \mathrm{Li}_2(x) \otimes 1 - \log\left(1-x\right) \otimes \log(x)$$

Symbol map

Coproduct & grading

The coproduct respects the grading of the algebra of MPLs

$$\Delta : \mathcal{H}_n \to \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q \,, \quad \Delta = \sum_{p+q=n} \Delta_{p,q}$$

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Iterative application

Applying the coproduct multiple times, we split up the MPLs as

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q \xrightarrow{\Delta \otimes \mathrm{id}} \bigoplus_{p+q+r=n} \mathcal{H}_p \otimes \mathcal{H}_q \otimes \mathcal{H}_r \xrightarrow{\Delta \otimes \mathrm{id} \otimes \mathrm{id}} \quad \cdots$$

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Symbol

The symbol is the $(1, \ldots, 1)$-component of the $(n-1)$-fold application of the coproduct

$$\mathcal{S} = \Delta_{1,\ldots,1} \mod \pi$$

$\mathcal{N} = 4$ pSYM loop amplitudes & cluster algebras

Key observation

The letters φ_{α_i} of $\mathcal{N} = 4$ pSYM loop amplitudes for $n = 6, 7$ are cluster $\mathcal{A}$-variables of the *cluster algebra* associated to $\mathrm{Gr}(4, n)$.
[Golden, Goncharov, Spradlin, Vergu, Volovich '13]

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Crucial input for cluster bootstrap:

1. "Guess" the collection of letters, the *alphabet*, of the amplitude
2. Construct the space of all weight $2L$ symbols
3. Fix amplitude with consistency conditions and physical constraints
4. "Integrate" the symbol to obtain the amplitude as a function

see eg. [2005.06735, 1108.4461, 1111.1704, 1308.2276, 1402.3300, 1407.4724, 1412.3763, 1612.08976, 1812.04640, ...]

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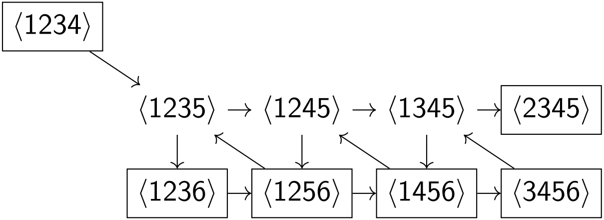
Cluster algebra structures at the level of the integrand were previously discovered by Arkani-Hamed et al in [1215.5605].

Outline

Grassmannian cluster algebras

$\mathrm{Gr}(4,6) \simeq A_3$



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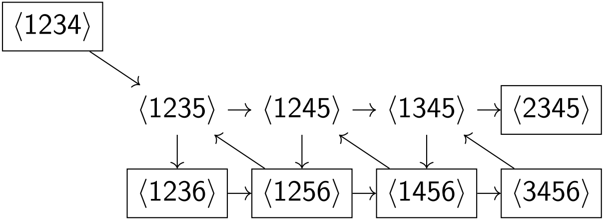
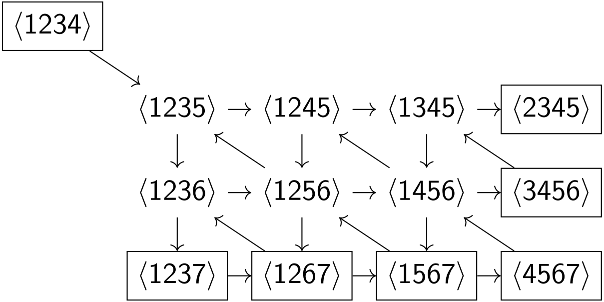


$\mathrm{Gr}(4,7) \simeq E_6$



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$$a_3 \longrightarrow a_4 \quad \xleftarrow{\mu_{a_2}} \quad a_2 \rightarrow \frac{1+a_2}{a_1}$$

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$a_2 \rightarrow \frac{1+a_2}{a_1}$

$\xleftarrow{\mu_{a_2}}$

$a_3 \longrightarrow a_4$

$\Big\uparrow \mu_{a_3}$

$a_5 \longleftarrow a_4$

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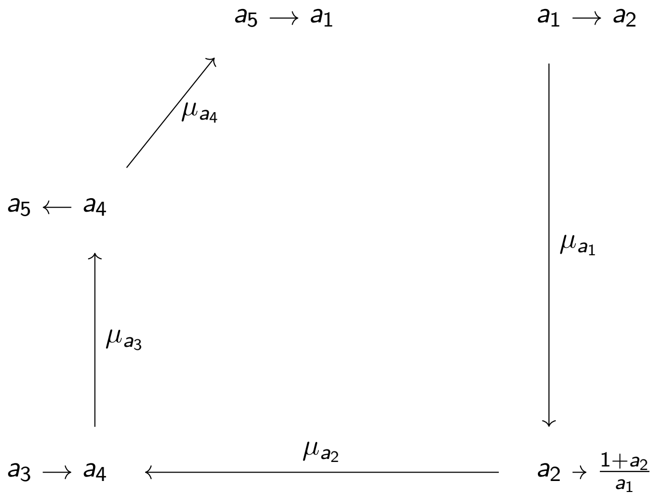

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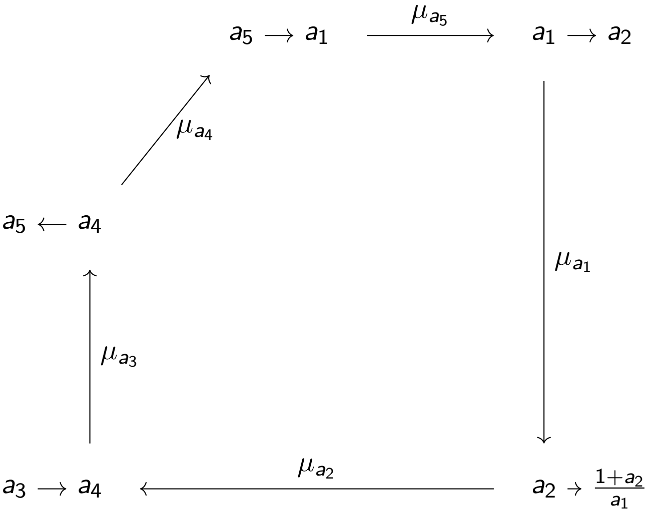
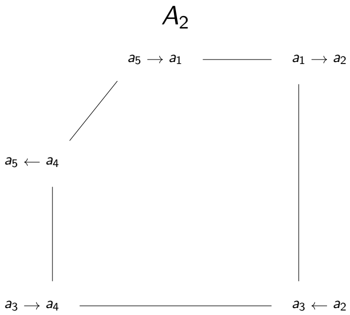
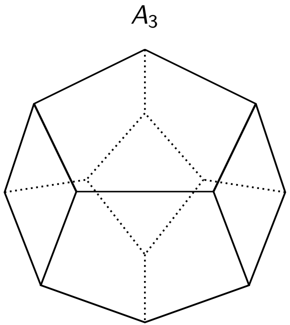


Cluster polytope





The *cluster polytope* is a geometric description of the cluster algebra, where

Vertex $\longleftrightarrow$ Cluster

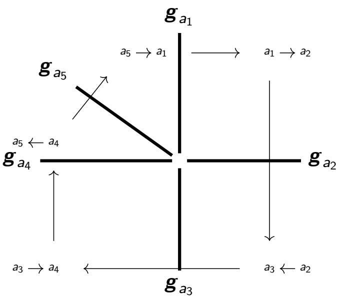
Edge $\longleftrightarrow$ Mutation / $(n-1)$-dim subalgebra

...

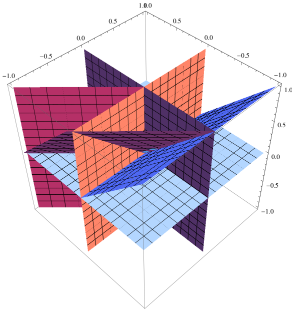
Facet $\longleftrightarrow$ Variable / 1-dim subalgebra

Cluster polytope & fan

A_2



A_3



Definition

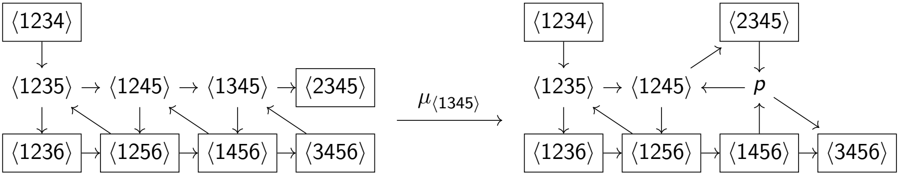
The normal fan to the cluster polytope is the *cluster fan*, with associations

$$\text{clusters} \longleftrightarrow \textit{cones}\,, \quad \text{variables} \longleftrightarrow \textit{rays}$$

see also [Fomin, Zelevinsky '01]

Outline

Cluster algebra of $\mathrm{Gr}(4,6)$



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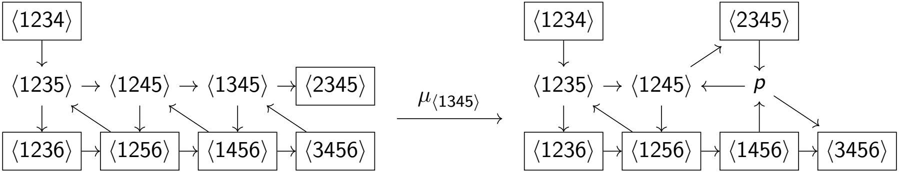

Using the mutation rules, the new variable p is given by

$$p = \frac{\langle 1245\rangle \langle 3456\rangle + \langle 1456\rangle \langle 2345\rangle}{\langle 1345\rangle}\,.$$

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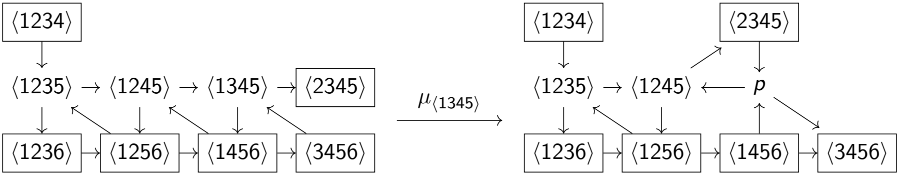

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However, the Plücker variables are related by *Plücker relations* such as

$$\langle 1245\rangle \langle 3456\rangle + \langle 1456\rangle \langle 2345\rangle - \langle 1345\rangle \langle 2456\rangle = 0$$

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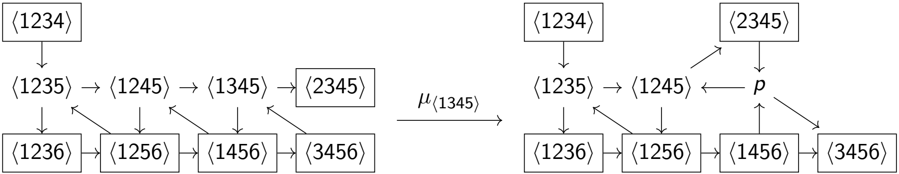

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$$\langle 1245\rangle \langle 3456\rangle + \langle 1456\rangle \langle 2345\rangle - \langle 1345\rangle \langle 2456\rangle = 0$$

so that we conclude that

$$p = \langle 2456\rangle \ .$$

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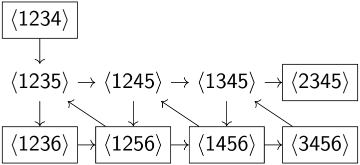

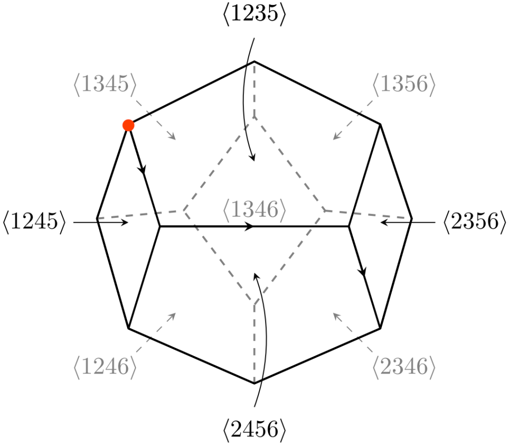

Cluster algebra of $\mathrm{Gr}(4,6)$

- The cluster variables of $\mathrm{Gr}(4,6)$ are the 14 Plücker variables, split into the 9 unfrozen variables

$$\{\langle 1235\rangle, \langle 1245\rangle, \langle 1345\rangle, \langle 1246\rangle, \langle 1346\rangle, \langle 1356\rangle, \langle 2346\rangle, \langle 2356\rangle, \langle 2456\rangle\}$$

and the 6 frozen variables

$$\{\langle 1234\rangle, \langle 1236\rangle, \langle 1256\rangle, \langle 1456\rangle, \langle 3456\rangle, \langle 2345\rangle\}$$

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$$\{\langle 1235\rangle, \langle 1245\rangle, \langle 1345\rangle, \langle 1246\rangle, \langle 1346\rangle, \langle 1356\rangle, \langle 2346\rangle, \langle 2356\rangle, \langle 2456\rangle\}$$

and the 6 frozen variables

$$\{\langle 1234\rangle, \langle 1236\rangle, \langle 1256\rangle, \langle 1456\rangle, \langle 3456\rangle, \langle 2345\rangle\}$$

- Out of these 15 variables, we can form the 9 dual conformally invariant cross ratios

$$\frac{\langle 1456\rangle\langle 1234\rangle}{\langle 1346\rangle\langle 1245\rangle}, \quad \frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle}, \quad \ldots$$

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Hexagon alphabet

The 9 DCI-invariant letters from $\mathrm{Gr}(4,6)$ precisely correspond to the alphabet known to describe six-particle amplitudes!

Steinmann relations & cluster adjacency

- With the knowledge of the hexagon alphabet, as well as physical constraints, the six-particle amplitude was bootstrapped with up to seven loops

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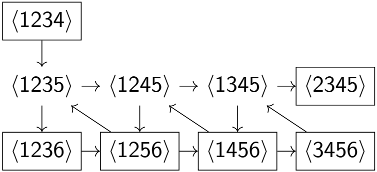
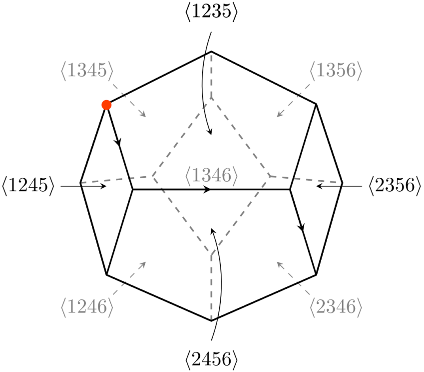
Cluster adjacency

Two letters can only appear next to each other in the symbol, if there exists a cluster that contains both of them. Geometrically, these pairs correspond to codimension-2 faces of the cluster polytope.
(see eg. [1710.10953, 1810.08149])

Cluster adjacency (example)





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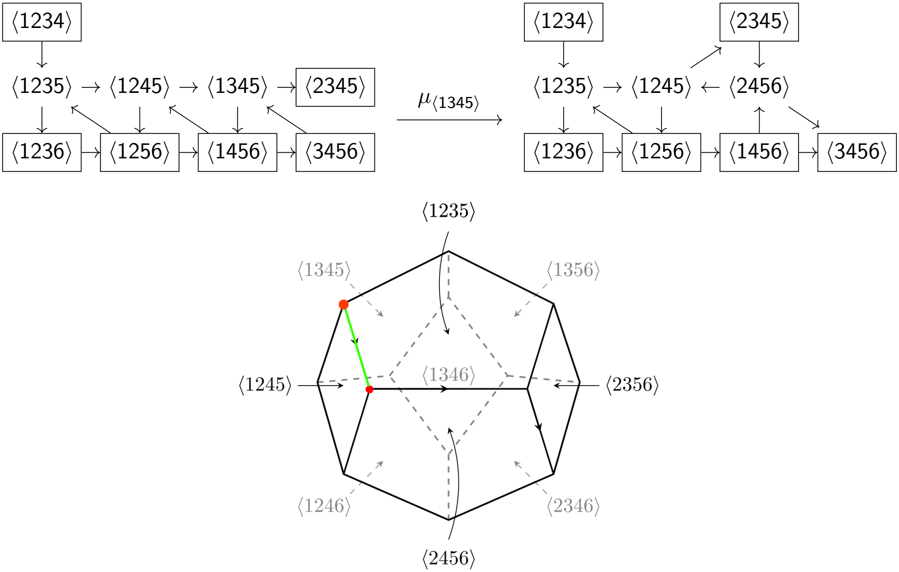
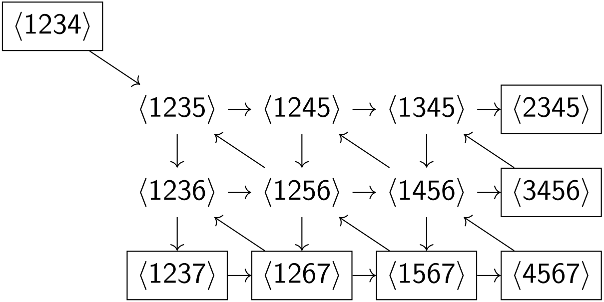


Heptagon alphabet

- Initial cluster of Gr(4, 7) cluster algebra is given by



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- Initial cluster of $\mathrm{Gr}(4,7)$ cluster algebra can be brought into the E_6 form

$$\begin{array}{ccccccccc} & & & & a_{13} & & & & \\ & & & & \uparrow & & & & \\ a_{24} & \longrightarrow & a_{51} & \longrightarrow & a_{62} & \longleftarrow & a_{41} & \longleftarrow & a_{33} \end{array}$$

see eg. [1412.3763, 1612.0876, 1812.04640, ...]

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- The (homogeneous) $\mathcal{A}$-coordinates contain the Plücker variables as well as bilinears such as

$$a_{61} = \frac{\langle 1356\rangle \langle 1247\rangle - \langle 1237\rangle \langle 1456\rangle}{\langle 1234\rangle \langle 1567\rangle}$$

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Heptagon alphabet

Together with the cyclic images a_{ij} obtained from the shifts $Z_m \to Z_{m+j-1}$, the alphabet of seven-particle scattering consists of the 42 $\mathcal{A}$-coordinates of the cluster algebra.

see eg. [1412.3763, 1612.0876, 1812.04640, ...]

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Long-standing open questions

1. Cluster algebras of $\mathrm{Gr}(4, n)$ become infinite for $n \geq 8 \implies$ no bootstrap
2. Cluster algebras cannot describe non-rational letters known to appear in alphabets for $n \geq 8$

Outline

Tropical geometry: definition

In essence

Tropical geometry is the algebraic geometry over the tropical semifield $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$, where

$$a \oplus b = \min(a, b) , \quad a \otimes b = a + b .$$

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Definition

The *tropical polynomial* Trg for some polynomial g is obtained by replacing addition with taking the minimum and multiplication by addition:

$$g = \sum_a c_a \cdot x_1^{a_1} \cdots x_n^{a_n} \longrightarrow \mathrm{Tr}g = \min_a \{a_1 \cdot x_1 + \cdots + a_n \cdot x_n\}$$

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Definition

The *tropical hypersurface* $V(\mathrm{Tr}g)$ is given by the points, where $\mathrm{Tr}g$ passes between regions of linearity. The *tropical variety* of a polynomial ideal is the intersection of all tropical hypersurfaces.

Tropical geometry: example

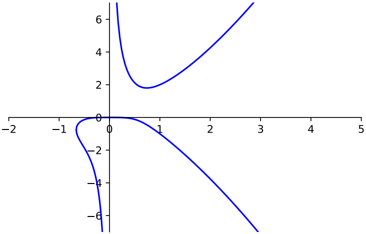
Tropicalise the polynomial g:

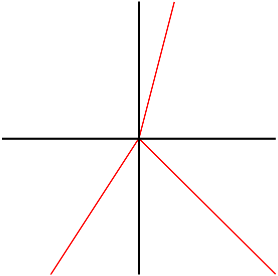
$$g(x, y) = 2x^3 + x^{-1}y - y^2 \longrightarrow \operatorname{Tr} g = \min(3x, -x + y, 2y)$$

The tropical hypersurface is given by the union of the equations:

$$3x = -x + y \leq 2y\,, \quad 3x = 2y \leq -x + y\,, \quad 2y = -x + y \leq 3x\ .$$

The affine variety of g on the left, the tropical variety on the right:





Kinematic space (revisited)

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$$p_{i_1,\ldots,i_k} = \det\left(Z_{i_1} \cdots Z_{i_k}\right) \,,$$

which obey the Plücker relations

$$p_{i_1\ldots i_r[i_{r+1}\ldots i_k}p_{j_1\ldots j_{r+1}]j_{r+2}\ldots j_k} = 0 \,.$$

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Definition

The *positive kinematic space* $\widetilde{\mathrm{Gr}}_+(k,n)$ is obtained by quotiening out rescalings of columns and restricting to $p_{i_1,\ldots,i_k} \geq 0$.

Parameterising the positive kinematic space

Theorem

$\widetilde{\mathrm{Gr}}_+(k,n)$ can be parameterised in terms of $(k-1)(n-k-1)$ independent parameters x_i by the web-parameterisation [Speyer, Williams '05].

Example

Consider for example $\widetilde{\mathrm{Gr}}_+(2,5)$, which is of dimension 2. The matrices $Z \in \mathrm{Gr}(2,5)$ can be parameterised as

$$Z = \begin{pmatrix} 1 & 0 & -1 & -1-x_1 & -1-x_1-x_1x_2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This implies a parameterisation for the Plücker variables, eg.

$$p_{25} = 1 + x_1 + x_1x_2\,.$$

Tropicalising the positive kinematic space

1. Start with the parameterisation of $\widetilde{\mathrm{Gr}}_+(k,n)$, e.g. for $\widetilde{\mathrm{Gr}}_+(2,5)$

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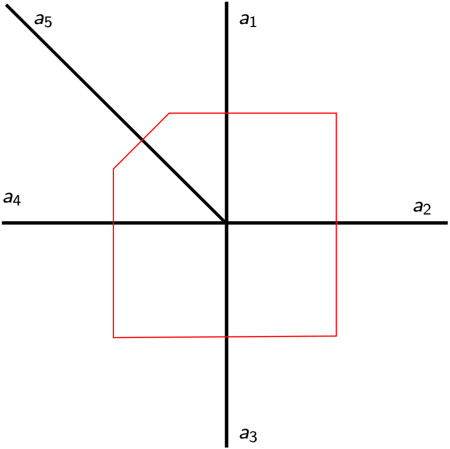
Tropical fan

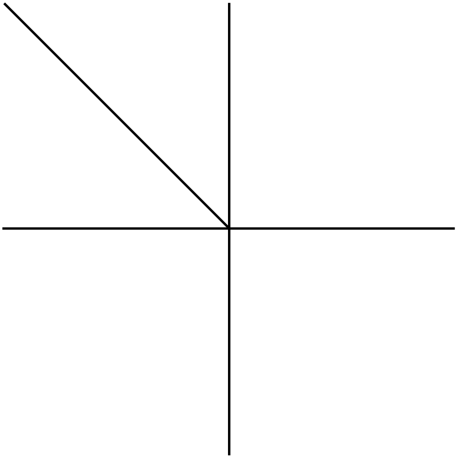
Tropicalising (a subset of) all Plücker variables gives the fan $(p)F_{k,n}$ of $(\mathrm{p})\widetilde{\mathrm{Tr}}_+(k,n)$, whose cones are the regions, where all tropicalised Plückers are linear.

Tropical kinematic space & cluster algebras

Cluster fan of $\mathrm{Gr}(2,5)$:



Fan $F_{2,5}$ of $\widetilde{\mathrm{Tr}}_+(2,5)$:



see eg. [Speyer, Williams '05], [Drummond, Foster, Gürdoğan, Kalousios '19]

Tropical kinematic space & cluster algebras

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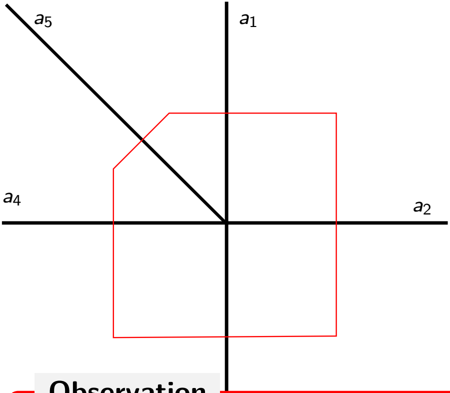

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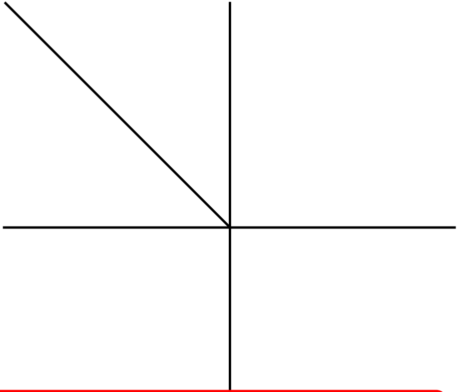

Observation

The cluster fan *triangulates* the tropical fan, in this case they are equivalent!

see eg. [Speyer, Williams '05], [Drummond, Foster, Gürdoğan, Kalousios '19]

Outline

Finite selection rule for infinite cluster algebras

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Cluster algebra of $\mathrm{Gr}(4, 8)$ predicts infinitely many singularities for $\mathcal{A}_8$.

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Revisiting the triangulation of $F_{k,n}$:

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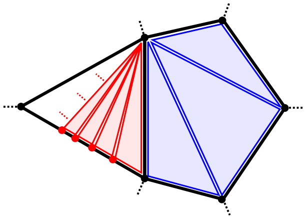
- This triangulation may contain redundant rays, infinitely triangulating an already triangular cone.



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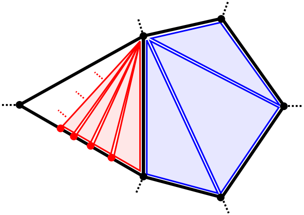

Selection rule

Stop mutating the cluster algebra, whenever you encounter a cluster containing redundant rays.

Rational alphabet of eight-particle amplitudes

- 70 (70 for partial tropicalisation) letters of degree one, the Plückers $\langle ijkl \rangle$,
- 120 (120) letters of degree two, e.g.

$$\langle 1457 \rangle \langle 2367 \rangle - \langle 1237 \rangle \langle 4567 \rangle \ ,$$

- 132 (90) letters of degree three, e.g.

$$\langle 1236 \rangle \langle 1578 \rangle \langle 3457 \rangle - \langle 1237 \rangle \langle 1578 \rangle \langle 3456 \rangle - \langle 1235 \rangle \langle 1678 \rangle \langle 3457 \rangle \ ,$$

- 32 (0) letters of degree four,
- 10 (0) letters of degree five.

Rational 8-particle alphabet

These form 356 (**272**) dual conformally invariant rational letters. All known rational letters appearing in eight-particle amplitudes included in the partial alphabet.

[NH, Papathanasiou '19], [Drummond, Foster, Gürdoan, Kalousious '19], [Arkani-Hamed, Lam, Spradlin '19], [Zhang, Li, He '19]

Rational alphabet of nine-particle amplitudes

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Finite subset

Using the same selection rule as before, with the partial tropicalisation

$$\widetilde{\mathrm{pTr}}_{+}(4,9): \text{ tropicalise } \{\langle i i+1 j j+1\rangle\,,\langle i j-1 j j+1\rangle\}\ ,$$

we obtain a finite subset with **3078** rational letters in **24,102,954** clusters.

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Rational 9-particle alphabet

Our proposed rational alphabet for 9-particle scattering amplitudes contains **3078** letters, which include the 531 rational letters known in the literature [He, Li, Zhang '20].

Outline

Square roots in the alphabet

Cluster variables

$\mathcal{A}$-variables of a cluster algebra are always rational functions of the initial variables (eg. Plücker variables $\langle ijkl \rangle$) due to the mutation relation

$$a_j \to a_j' = \frac{\prod_{i\to j} a_i + \prod_{j\to i} a_i}{a_j}$$

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Square-root letters

Cluster algebras hence cannot describe square-root letters, such as $\sqrt{\Delta_{ijkl}}$ whereas

$$\Delta_{ijkl} = (f_{ij} f_{kl} - f_{ik} f_{jl} - f_{il} f_{jk})^2 - 4 f_{ij} f_{kl} f_{kl} f_{il} \,,$$
$$f_{ij} = \langle i i+1 j j+1 \rangle$$

which is known to appear in the symbol of eight-particle amplitudes!
(see eg. [He, Li, Zhang '19'20; Li, Zhang '21])

Infinite mutation sequences & square-root letters

For the infinite mutation sequence of the affine $A_1^{(1)}$ cluster algebra

$$a_1 \rightrightarrows a_2 \quad \xrightarrow{\mu_1} \quad a_3 \leftleftarrows a_2 \quad \xrightarrow{\mu_2} \quad a_3 \rightrightarrows a_4 \quad \xrightarrow{\mu_2} \quad \cdots$$

we get a recursion relation for the variables a_i, which can be solved for

$$\lim \frac{a_i}{a_{i-1}} = \frac{a_2}{2a_1}\left(1 + x_1 + x_1 x_2 + \sqrt{(1 + x_1 + x_1 x_2)^2 - 4x_1 x_2}\right),$$

where $x_1 = 1/a_2^2, x_2 = a_1^2$. [Canakci, Schiffler '16]

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Square-root letters

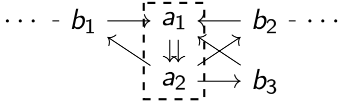
Including (the limit of) infinite mutation sequences, we obtain generalised cluster variables that correspond to the square-root letters of the amplitude.

(see eg. [NH, Papathanasiou '19],[Arkani-Hamed, Lam, Spradlin '19],[Drummond, Foster, Gürdoğan, Kalousios '19])

Algebraic alphabet of eight-particle amplitudes

Origin clusters

In the cluster algebra of $\mathrm{Gr}(4,8)$ truncated by $\widetilde{\mathrm{pTr}}_+(4,8)$, we find 3200 $A_1^{(1)}$ *origin clusters* of the following type



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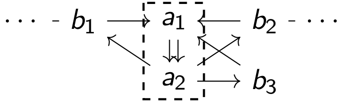

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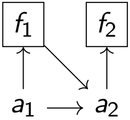
Algebraic 8-particle alphabet

The 64 origin clusters give rise to 128 algebraic letters which can be reduced to the **18 multiplicatively independent algebraic letters** previously found.

[NH, Papathanasiou '19], [Drummond, Foster, Gürdoan, Kalousious '19], [Zhang, Li, He '19]

Revisiting frozen variables

Consider the following A_2 cluster algebra with two frozen variables f_1, f_2.



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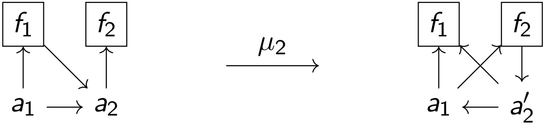

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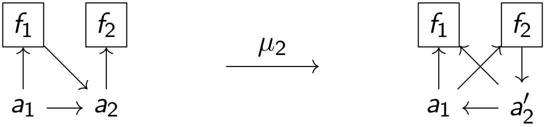

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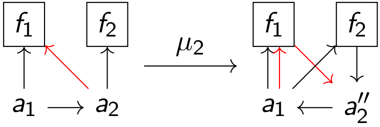

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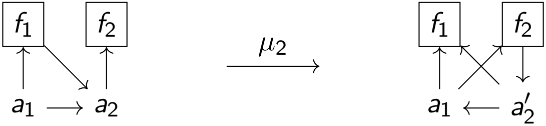

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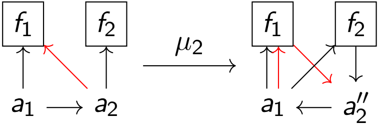

$$a_2'' = \frac{a_1 + f_1 f_2}{a_2}$$

$\Rightarrow$ Using so-called *coefficients* instead of frozen variables solves this issue!

Cluster algebras with coefficients

Coefficients

For a cluster with variables $a_1, \ldots, a_n$ and frozen variables $f_{n+1}, \ldots, f_{n+m}$, and adjacency matrix b_{ij}, the *coefficients* $y_1, \ldots, y_n$ are

$$y_i = \prod_{j=n+1}^{n+m} f_j^{b_{ji}} \,.$$

Mutation (coefficients)

When mutating at node j, coefficients mutate as

$$y_j \to y_j^{-1}\,, \quad y_i \to y_i y_j^{\max(0,b_{ji})} \left(1 \hat{\oplus} y_j\right)^{-b_{ji}} \text{ if } i \neq j$$

where the *cluster-tropical addition* is defined on the frozen variables as

$$\prod_i (f_i)^{b_i} \hat{\oplus} \prod_j (f_j)^{c_j} = \prod_i (f_i)^{\min(b_i, c_i)}$$

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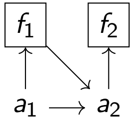
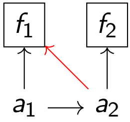
Mutation ($\mathcal{A}$-variables)

The mutation rule of the $\mathcal{A}$-variables under a mutation at node j is changed to

$$a_j = \frac{y_j \prod_{i\to j} a_i^{b_{ij}} + \prod_{j\to i} a_i^{b_{ij}}}{\left(1 \hat{\oplus} y_j\right) a_j}$$

Cluster algebras with coefficients (example)





The coefficients of the initial cluster are given by

$$y_1 = f_1^{-1}\,, \quad y_2 = f_1 f_2^{-1}\,, \qquad\qquad y_1 = f_1^{-1}\,, \quad y_2 = f_1^{-1} f_2^{-1}\,,$$
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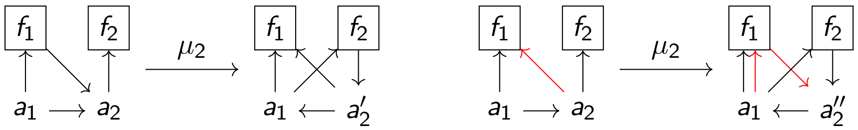

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$$y_1' = y_1(1 \hat{\oplus} y_2) = f_1^{-1} f_2^{-1}, \qquad\qquad y_1' = y_1(1 \hat{\oplus} y_2) = f_1^{-1},$$

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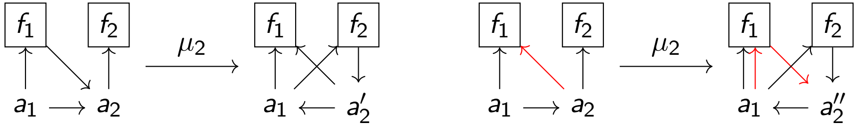

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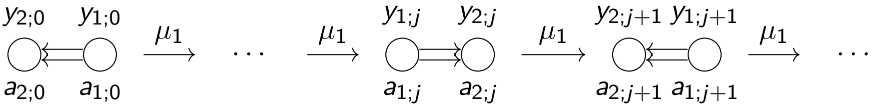
and the new $\mathcal{A}$-variable as

$$a_2' = \frac{y_2 a_1 + 1}{a_2\,(1\hat{\oplus} y_2)} = \frac{f_1 a_1 + f_2}{a_2} \qquad\qquad a_2' = \frac{y_2 a_1 + 1}{a_2\,(1\hat{\oplus} y_2)} = \frac{a_1 + f_1 f_2}{a_2}$$

Affine rank-2 cluster algebra $A_1^{(1)}$



Recursion relation

The variables and coefficients along the sequence are related by

$$a_{2;j+1} = \frac{a_{2;j}^2}{a_{1;j}} \frac{1 + x_{1;j}}{1 \hat{\oplus} y_{1;j}} \,, \quad a_{1;j+1} = a_{2;j} \,,$$

$$y_{1;j+1} = \frac{y_{2;j}\, y_{1;j}^2}{\left(1 \hat{\oplus} y_{1;j}\right)^2} \,, \quad y_{2;j+1} = \left(y_{1;j}\right)^{-1} \,.$$

We also introduce the auxiliary sequence

$$\gamma_j = 1 \hat{\oplus} y_{1;j} \hat{\oplus} y_{1;j} \left(y_{1;j-1}\right)^{-1} \,.$$

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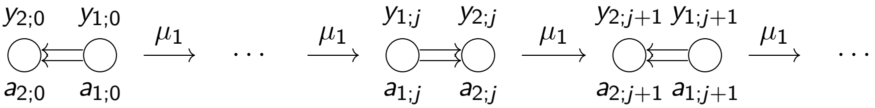

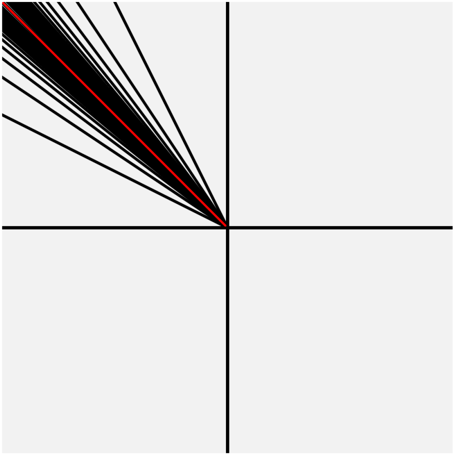

cf. [NH, Papathanasiou '19], [Canakci, Schiffler '16], [Reading '17], [Fordy, Marsh '09]

Invariants and solution of $A_1^{(1)}$

Invariants

Along the sequence, the following two quantities are invariant

$$K_1 = \left(\gamma_0\gamma_j^{-1}\beta_0^{-1}\beta_j\right)\left[1 + x_{1;j} + x_{1;j}(x_{1;j-1})^{-1}\right] = 1 + x_{1;0} + x_{1;0}x_{2;0}\,,$$

$$K_2 = \left(\gamma_0\gamma_j^{-1}\beta_0^{-1}\beta_j\right)^2\left[x_{1;j}(x_{1;j-1})^{-1}\right] = x_{1;0}x_{2;0}\,,$$

with respect to the two $\mathcal{X}$-variables x_1, x_2 of $A_1^{(1)}$ within the *origin cluster*.

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Solution

The recursion relation is solved for $a_{1;j}$ by

$$a_{1;j} = (\gamma_0 \cdots \gamma_{j-1})\left[C_+(\beta_+)^j + C_-(\beta_-)^j\right]\,,$$
$$C_\pm = a_{1;0}\frac{\pm 2 \mp K_1 + \sqrt{K_1^2 - 4K_2}}{2\sqrt{K_1^2 - 4K_2}}\,,\ \beta_\pm = \frac{a_{2;0}}{a_{1;0}}\frac{K_1 \pm \sqrt{K_1^2 - 4K_2}}{2\gamma_0}\,.$$

Infinite mutation sequences & square-root letters

Algebraic letters

Given some infinite $A_1^{(1)}$ mutation sequence, we associate to it the two algebraic letters

$$\phi_1 = \frac{C_+}{C_-} = \frac{K_1 - 2 + \sqrt{K_1^2 - 4K_2}}{K_1 - 2 - \sqrt{K_1^2 - 4K_2}}\,,$$

$$\phi_2 = \frac{\tilde{C}_+}{\tilde{C}_-} = \frac{K_1 - 2K_2 + \sqrt{K_1^2 - 4K_2}}{K_1 - 2K_2 - \sqrt{K_1^2 - 4K_2}}\,.$$

Nine-particle alphabet: new obstructions

Rational alphabet

1. Much more complicated rational letters, with some being polynomials with tens of thousands of terms.
2. Bootstrap becomes infeasible, as the linear spaces the approach is based on are of size $(\#\text{ letters})^{2L}$.

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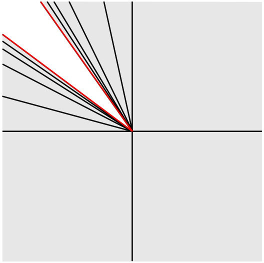
Algebraic alphabet

1. Infinite sequences of the type $A_1^{(1)}$ yield 324 rays in addition to the rational rays.
2. 27 rays of $\widetilde{\text{pTr}}_+(4,9)$ are not accessible from the cluster fan.
3. The cluster fan now has higher-dimensional "holes".



Outline

1. Scattering amplitudes, multiple polylogarithms and symbols
2. Cluster algebras
3. Six- and seven-particle amplitudes
4. Tropical cluster algebras
5. Eight-particle & nine-particle amplitudes
6. Infinite mutation sequences
7. Conclusions & Outlook
8. Bonus material

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- The relation to tropical Grassmannians provides a finite selection rule
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Some questions remain:

- Which geometry actually describes loop amplitudes of $\mathcal{N} = 4$ pSYM?
- How can bootstrapping the amplitudes made feasible?
- Is there a way to access the 27 still missing rays?

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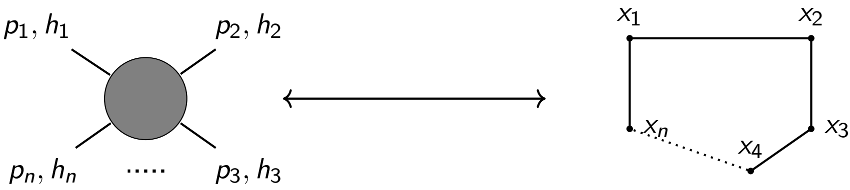
Kinematics of $\mathcal{N} = 4$ pSYM amplitudes



Definition

Instead of the massless momenta p_i, parameterise the kinematics in terms of the *dual variables* $x_i \in \mathbb{R}^{1,3}$ defined by

$$x_i - x_{i+1} = p_i \, .$$

Dual conformal symmetry

A *hidden symmetry* of the theory, which is not present on the level of the Lagrangian, that acts on the dual variables and reduces their degrees of freedom. See eg. [0807.1095, 1012.4002].

Momentum twistors

1. Represent the dual variables $x_i \in \mathbb{R}^{1,3}$ as projective null vectors $X^M \in \mathbb{R}^{2,4}$, with $X^2 = 0$, $X \sim \lambda X$.

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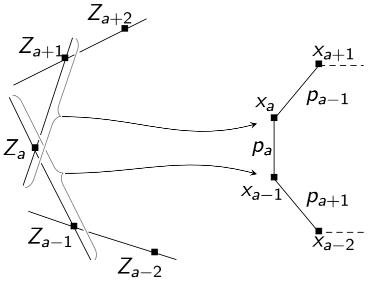

see eg.
[0905.1473, 1012.6032]

Cluster algebras: definition

Clusters

A cluster algebra of rank r consists of clusters

$$\Sigma = ((a_1, \dots, a_r), (y_1, \dots, y_r), Q) \ ,$$

containing the *cluster $\mathcal{A}$-variables* a_i, their *coefficients* y_i and the quiver Q with adjacency matrix b, encoding the connectivity of the variables.

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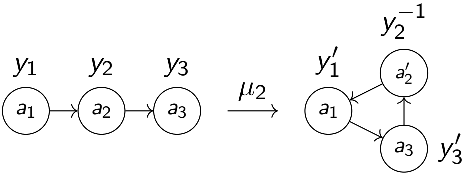
Mutation

Mutation takes a cluster into another cluster, $\mu_j : \Sigma \to \Sigma'$, eg.



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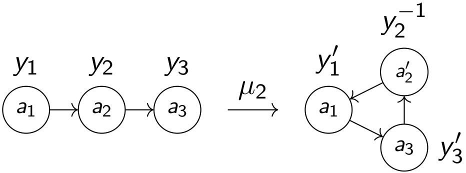

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$$a_2 \to a_2' = \frac{y_2 a_1 + a_3}{a_2\left(1 \hat{\oplus} y_2\right)}\,, \quad y_1 \to y_1' = y_1(1 \hat{\oplus} y_2)\,, \quad y_3 \to y_3' = y_3 \frac{y_2}{1 \hat{\oplus} y_2}\,,$$

where $\hat{\oplus}$ is an addition on the field of coefficients.

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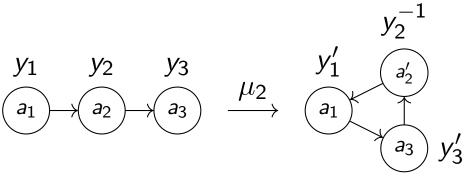

$\mathcal{X}$-variables

To each node in a cluster, we also associate a $\mathcal{X}$-*variable*, given by eg.

$$x_1 = \frac{1}{a_2} \cdot y_1 \,, \quad x_2 = \frac{a_1}{a_3} \cdot y_2 \,, \quad x_3 = a_2 \cdot y_3 \,.$$

based on [Fomin, Zelevinsky '06]

Hexagon alphabet

- The symbols known to appear in six-particle amplitudes (the *hexagon alphabet*) are functions of the cross ratios

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,, v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}\,, w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{25}^2}\,,$$

where $x_{ij}^2 = (x_i - x_j)^2$ are Lorentz-invariant distances of dual variables.

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- The entire alphabet consists of the nine letters

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where the variables y_u, y_v, y_w are defined as

$$y_u = \frac{u - z_+}{u - z_-},$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right], \quad \Delta = (1-u-v-w)^2 - 4uvw$$

see eg. [1108.4461, 1111.1704, 1308.2276, ...]

Hexagon alphabet (cont'd)

- When parameterised in terms of the momentum twistors $Z_1, \ldots, Z_6$, the cross ratios are given by

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- Also, in terms of the momentum twistors, Δ is a perfect square given by

$$\sqrt{\Delta} = \pm \frac{\langle 3456\rangle \langle 1256\rangle \langle 1234\rangle + \langle 1456\rangle \langle 1236\rangle \langle 2345\rangle}{\langle 2356\rangle \langle 1346\rangle \langle 1245\rangle},$$

whereas the two signs are related by the spacetime parity transformation, ie inversion of the spatial components of the momenta $(p_i)^k \to -(p_i)^k$

Periodic clusters and sequences

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Recursive sequence

Due to periodicity, we can repeat the mutation infinite times and thus get sequences $(a_i)_{i\in\mathbb{N}}$ of cluster variables and $(y_i)_{i\in\mathbb{N}}$ of coefficients with the same mutation rule at each position $i \in \mathbb{N}$, ie. a *recursion relation*.

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A quiver Q is said to be *cluster mutation-periodic* of period p, if there is a sequence of p mutations that results in a quiver isomorphic to Q.

Recursive sequence

Due to periodicity, we can repeat the mutation infinite times and thus get sequences $(a_i)_{i\in\mathbb{N}}$ of cluster variables and $(y_i)_{i\in\mathbb{N}}$ of coefficients with the same mutation rule at each position $i \in \mathbb{N}$, ie. a *recursion relation*.

Example

Consider the affine rank-2 cluster algebra of $A_1^{(1)}$ Dynkin type. After mutating any of its nodes, the quiver is the same as before up to switching the labels of the nodes: