Scattering Amplitudes– symbols and cluster algebras

Felix Tellander

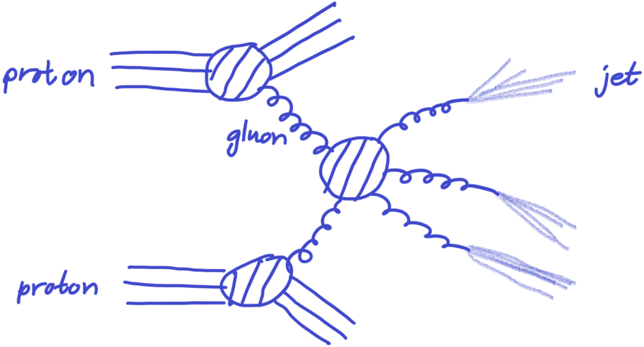
November 18, 2021

Overview

- Scattering amplitudes – planar $\mathcal{N} = 4$ super Yang-Mills theory
- Focus on 6 and 7 particle amplitudes
- What are the right functions?
- What are the right variables?
- Connection to cluster algebras

Amplitudes

A collision at the Large Hadron Collider (LHC) at CERN.



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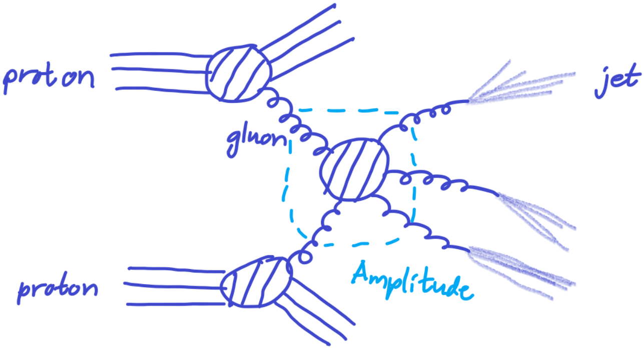

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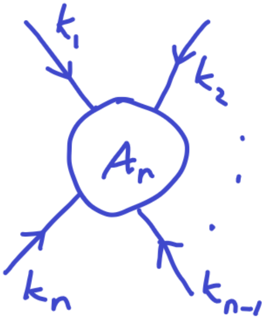
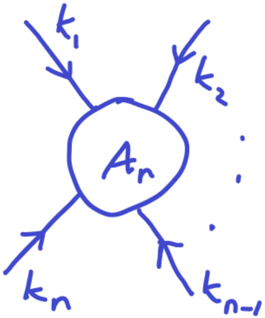
Amplitudes

Generalized Polyogarithms

Twistors

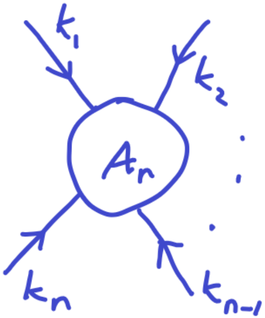
Cluster Algebras





Depends on:

- Momenta: $k_1, \ldots, k_n \in \mathbb{R}^{1,3}$
- Helicities: $h_1, \ldots, h_n \in \{-1, +1\}$
- $SU(N)$ generators $T^{a_1}, \ldots, T^{a_{N^2-1}}$



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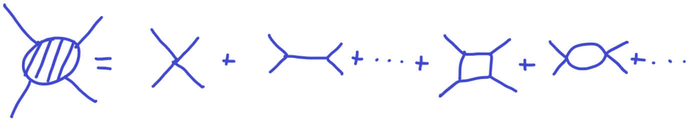
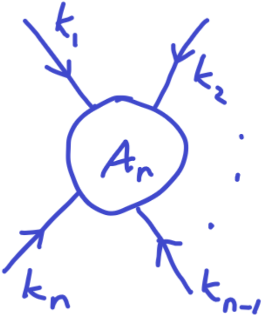
Expansion in terms of Feynman diagrams:





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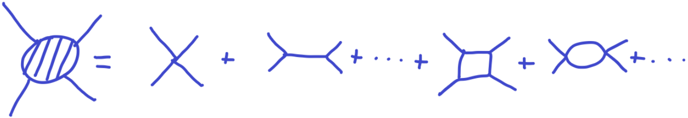

Contribution at each loop order grows factorially.
The amplitude program aims to calculate amplitudes without having to evaluate and sum thousands of Feynman diagrams.

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Can be collected into one superfield:

$$
\begin{aligned}
\Phi =& G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\overline{\Gamma}^D \\
&+ \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-
\end{aligned}
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Colour Management

't Hooft planar limit: send the number of colours $N \to \infty$ s.t. $g^2 N$ is constant.

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One single trace of $SU(N)$ generators become dominant for n-gluon amplitudes:

$$\mathcal{A}_n^{L-\mathrm{loop}}(\{k_i, h_i, a_i\}) \sim \sum_{\sigma \in S_n/cyclic} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n^{(L)}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n}))$$

$A_n^{(L)}$ is the colour-ordered amplitude, it no longer depends on T^a.

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Momenta can be expressed in dual coordinates: $k_i = x_{i+1} - x_i$.
Dual conformal invariance: $x_i^\mu \to x_i^\mu / x_i^2$

Generalized Polyogarithms

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Definition 1 (GPL)

Let $n\geq 0$ be an integer, then the *generalised polylogarithm* (GPL) is the iterated integral

$$G(a_1,\ldots,a_n;z)=\int_0^z\frac{G(a_2,\ldots,a_n;t)}{t-a_1}dt \tag{1}$$

with $G(z)=G(;z)=1$ and $a_1,\ldots,a_n,z\in\mathbb{C}$.

Example:

$$G(0,1;z) = \int_0^z \frac{G(1;t_1)}{t_1} dt_1 = \int_0^z dt_1 \int_0^{t_1} dt_2 \frac{1}{t_1(t_2-1)} = -\mathrm{Li}_2(z)$$

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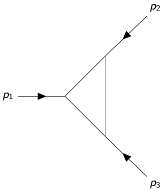
Example (Off-shell triangle in dimensional regularisation):

$$T(p_1^2, p_2^2, p_3^2) :=$$



$$= e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2}$$

$$T(p_1^2, p_2^2, p_3^2) = \frac{2}{\sqrt{\lambda}} \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(z\bar{z}) \log\left(\frac{1-z}{1-\bar{z}} \right) \right] + \mathcal{O}(\epsilon)$$

where

$$D = 4 - 2\epsilon,\ \ \gamma_E = -\Gamma'(1),\ \ p_1^2/p_3^2 = z\bar{z},\ \ p_2^2/p_3^2 = (1-z)(1-\bar{z})$$
$$\lambda := \lambda(p_1^2, p_2^2, p_3^2),\ \ \lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

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The number n is called the *weight* of the GPL.

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Something stronger is true:

Theorem 3 (Hopf algebra)

Generalised polylogarithms form a Hopf algebra.

Think:

- Family of algebraic objects (GPLs, graphs, posets, matroids,...)
- Rules for "*merging*" and "*breaking*" these objects

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Last term: 2-fold tensor product with element of weight one.
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Definition 4 (Symbol)

Let f_n be a GPL of weight n. Then the *symbol* $\mathcal{S}(f_k)$ is the sum

$$\mathcal{S}(f_n) := \sum_{i_1,\ldots,i_n} f_0^{(i_1,\ldots,i_n)} (\log \alpha_{i_1} \otimes \cdots \otimes \log \alpha_{i_n}) \tag{2}$$

where $f_0^{(i_1,\ldots,i_n)}$ are rational (i.e. of weight zero).
We call the collection of all α_{i_j} the *symbol alphabet*.

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Empirical evidence that colour-ordered L-loop amplitude can be expressed as GPLs of weight $2L$.

Momentum Twistors

- Dual space-time variables $x^\mu \in \mathbb{R}^{1,3}$ can be represented as:

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- Mandelstam variables:

$$(x - x')^2 \propto \epsilon_{IJKL} Z^I \tilde{Z}^J Z'^K \tilde{Z}'^L := \left\langle Z \tilde{Z} Z' \tilde{Z}' \right\rangle$$

Configuration Space

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The configuration space can be realized as the Grassmannian

$$\mathrm{Conf}_n(\mathbb{CP}^3) = \frac{\mathrm{Gr}(4,n)}{(\mathbb{C}^*)^{n-1}}$$

This is a collection of n ordered momentum twistors on $\mathbb{CP}^3$.

Cluster Algebras

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Example: A_2 cluster algebra

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- Initial cluster: $\{a_1, a_2\}$
- Clusters: $\{a_m, a_{m+1}\}$
- Mutation:
  $\{a_{m-1}, a_m\} \to \{a_m, a_{m+1}\} \ : \ a_{m-1} \mapsto a_{m+1} = (1 + a_m)/a_{m-1}$

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Finite number of cluster variables:

$$a_3 = \frac{1 + a_2}{a_1},\ a_4 = \frac{1 + a_1 + a_2}{a_1 a_2},\ a_5 = \frac{1 + a_1}{a_2},\ a_6 = a_1,\ a_7 = a_2$$

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The coordinates we used in A_2 example are $\mathcal{A}$-coordinates.

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where $[x]_+ = \max(0, x)$ and

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The cluster $\mathcal{X}$-coordinates are defined as

$$x_i := \prod_j a_j^{b_{ji}}$$

and mutate as

$$x'_i = \begin{cases} 1/x_i & i = k \\ x_i(1 + x_k^{\operatorname{sgn}(b_{ki})})^{-b_{ki}} & i \neq k \end{cases}$$

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Example $(\mathrm{Gr}(2,5))$: The Grassmannian $\mathrm{Gr}(2,5)$ is associated to the configuration space of 5 points on $\mathbb{CP}^1$.
Initial quiver:

$$
\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
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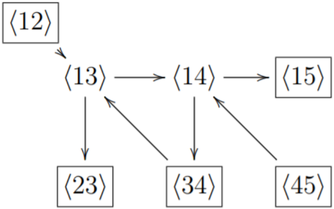

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$$x_k = \frac{\prod_{i:i\to k} a_i}{\prod_{j:k\to j} a_j}$$

$$x_{13} = \frac{\langle 12\rangle \langle 34\rangle}{\langle 23\rangle \langle 14\rangle}, \quad x_{14} = \frac{\langle 13\rangle \langle 45\rangle}{\langle 15\rangle \langle 34\rangle}$$

The simplest examples in planar $\mathcal{N}=4$ SYM are for $n=6,\ 7$ external particles, i.e. projective configurations of 6 resp. 7 points in $\mathbb{CP}^3$. The principal part of the quiver for $\mathrm{Gr}(4,6)$ is the A_3 quiver.

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We know that the A_{d-3} cluster algebra can be realized as a quiver from triangulating a d-gon which is the same as the cluster algebra from a $\mathrm{Gr}(2, d)$. Grassmann duality

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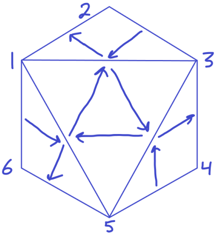
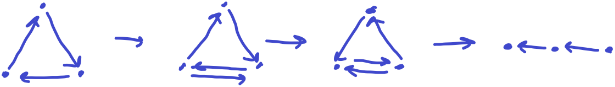


$\mathcal{X}$-coordinates:

$$\frac{\langle 23\rangle\langle 15\rangle}{\langle 12\rangle\langle 35\rangle},\ \frac{\langle 45\rangle\langle 13\rangle}{\langle 34\rangle\langle 15\rangle},\ \frac{\langle 35\rangle\langle 16\rangle}{\langle 13\rangle\langle 56\rangle}$$

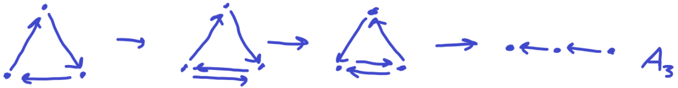
Dual coordinates (twistors):
$\frac{\langle 1456\rangle\langle 2346\rangle}{\langle 3456\rangle\langle 1246\rangle}$ etc.

Principal quiver:

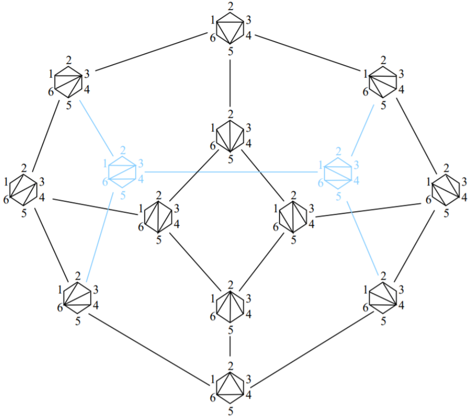


A_3

Principal quiver:



14 different quivers can be collected in the third associahedron:



2-loop MHV amplitude for $n = 6$ is known explicitly:

$$R_6^{(2)} = \frac{-1}{2}\sum_{i=1}^{3} \mathrm{Li}_4(-v_i) + \cdots$$

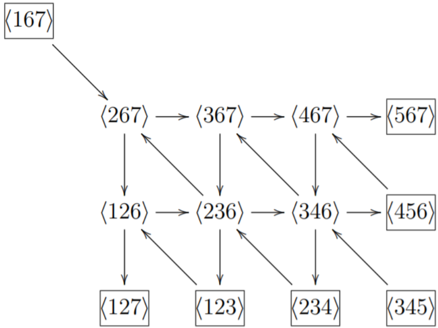
where

$$v_1 = \frac{\langle 35\rangle \langle 26\rangle}{\langle 23\rangle \langle 56\rangle}, \quad v_2 = \frac{\langle 13\rangle \langle 46\rangle}{\langle 16\rangle \langle 34\rangle}, \quad v_3 = \frac{\langle 15\rangle \langle 24\rangle}{\langle 45\rangle \langle 12\rangle}$$

these all appear ass $\mathcal{X}$-coordinates of $\mathrm{Gr}(2,6) \simeq \mathrm{Gr}(4,6)$ cluster algebra.

Only 9 of the available 15 $\mathcal{X}$-coordinates appear in $R_6^{(2)}$.

$\mathrm{Gr}(4,7)$:



After 9 mutations the principal part of this quiver becomes the E_6 quiver.

$\mathrm{Gr}(4,6)$ and $\mathrm{Gr}(4,7)$ are both finite cluster algebras, starting at $n \geq 8$ we get infinite algebras.