ZMP Seminar, Winter 2021/2022

Introduction to cluster Algebras II

Murad Alim

Based on:
Reviews by Keller
Vergu
L. Williams

Plan:

1- Reminder

2- Introduction & motivation

3- Cluster algebras of geometric type

4- What are Grassmanians?

5- Quivers & Cluster algebras of Grassmanians

1- Reminder:

Last time we introduced quivers

$$Q = (\underline{Q_0}, \underline{Q_1}, \underline{s}, \underline{t})$$

vertices, arrows, source, target

e.g. A_2 quiver

$$1 \longrightarrow 2$$

Skew symmetric matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\underline{b_{ij}}$ = # arrows from i to j − # arrows from j to i mutation : $1 \leftarrow 2$

Seed mutation, cluster algebras

$$\boxed{\mathcal{F} := \mathbb{Q}(x_1, x_2)} \qquad 1 \longrightarrow 2$$

a seed is a pair $(\underset{\substack{\nearrow \\ \text{Quiver}}}{\underline{R}}, \underset{\substack{\uparrow \\ \text{sequence of gen. of } \mathcal{F}}}{\underline{u}})$

Ex. $(1 \longrightarrow 2, \{x_1, x_2\})$

$$\mu_1((R,u)) = (R,u)' \; (1 \longleftarrow 2, \{\underbrace{\frac{1+x_2}{x_1}}_{x_3}, x_2\})$$

$$\mu_2((R,u)') = (1 \longrightarrow 2, \{\frac{1+x_2}{x_1}, \underbrace{\frac{1+x_1+x_2}{x\,x_2}}_{x_4}\})$$

$$\mu_1((R,u)'') = (1 \longleftarrow 2, \{\underbrace{\frac{1+x_1}{x_2}}_{x_5}, \frac{1+x_1+x_2}{x_1x_2}\})$$

$$\mu_2((R,u)''') = (1 \longrightarrow 2, \{\frac{1+x_1}{x_2}, \underset{\substack{\| \\ x_6}}{x_1}\})$$

$$\mu_1((R,u)'''') = \{1 \longleftarrow 2, \{\underset{\substack{\| \\ x_7}}{x_2}, x_1\}))$$

Here: The clusle algebra $\mathcal{A}_{A_2}$ is generated as a $\mathbb{Q}$-algebra by the cluster variables x_m, $m \in \mathbb{Z}$, submitted to the exchange relations

$$\boxed{x_{m-1} x_{m+1} = 1 + x_m,} \quad m \in \mathbb{Z}$$

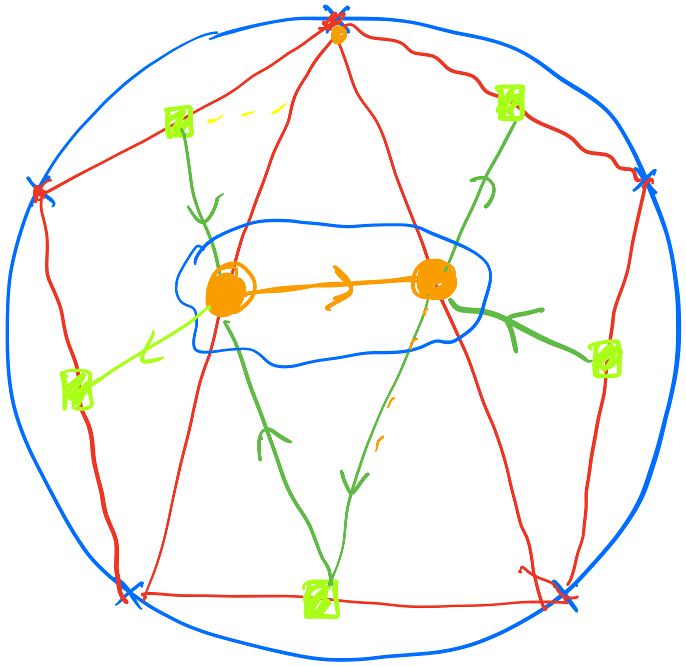
Its Clusters are pairs of consecutive variables $\{x_m, x_{m+1}\}$, $m \in \mathbb{Z}$.

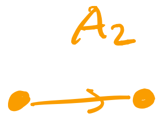
- * The initial cluster is $\{x_1, x_2\}$ and two clusters are linked by mutation iff they share exactly one variable.

- * The exchange relations allow one to write each cluster variable as a rational expression in the initial variables x_1, x_2 and thus to identify $\mathcal{A}_{A_2}$ with a subalgebra of the field $\mathbb{Q}(x_1, x_2)$.

2. Introduction & Motivation :

Consider the triangulation of a disk with 5 punctures on the boundary:

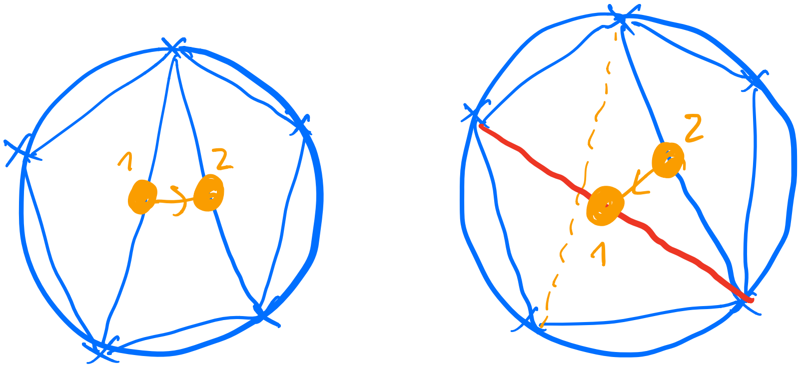




Assign a quiver to a given triangulation by the following rules:

- * assign a node ● (🟩) to each internal (external) arc of the triangulation.
- * assign an arrow to each pair of nodes which share a triangle using an orientation convention

mutation on node 1



mutation corresponds to flipping the diagonal within a quadrilateral

Remark: mutation in this context corresponds to a flip of the triangulation.

Adjacency matrix:

principal part B =

$$\tilde{B} = \left(\begin{array}{cc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & & & & & \\ 1 & -1 & & & & & \\ -1 & 0 & & & 0 & & \\ 1 & 0 & & & & & \\ 0 & -1 & & & & & \end{array}\right)$$

3. Cluster algebras of geometric type:

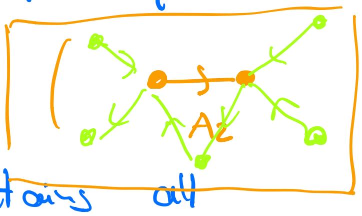
2, 7

3.1 Definition: Let $1 \leq k \leq n$ be integers.

Let $\tilde{Q}$ be an ice quiver of type (k,n), i.e. a quiver with n vertices which does not have any arrows between vertices i, j which are both strictly greater than k.

ex: Triangulation

The principal part of $\tilde{Q}$ is the full subquiver Q whose vertices are $1, \dots, k$



(full if with any two vertices, it contains all the arrows linking them)

ex:

The vertices $k+1, \dots, n$ are called frozen vertices.

The cluster algebra associated to the ice quiver $\tilde{Q}$

$$\boxed{\mathcal{A}_{\tilde{Q}}} \subset \mathbb{Q}(x_1, \dots, x_n) \text{ is defined}$$

in the same manner as the cluster algebra associated with a quiver but:

- only mutations w.r.t. non frozen vertices are allowed and no arrows between frozen vertices are added in the mutation.
- the variables $\underline{x_{k+1}, \dots, x_n}$ which belong to all clusters are called coefficients rather than cluster variables
- the cluster type of the ice quiver is that of its principal part.

3.2| Rem. The datum of $\widetilde{Q}$ is equivalent to that of the $n \times k$ matrix $\widetilde{B}$

where b_{ij} = # arrows from $i \to j$ − # arrows from $j \to i$

$1 \le i \le n$, $1 \le j \le k$.

The top $k \times k$ part B of $\widetilde{B}$ is called its principal part.

3.3| Theorem FZ '03

Each cluster variable in $\mathcal{A}_{\widetilde{Q}}$ is a Laurent polynomial in the initial variables $x_1, \dots, x_k$ (x_1, x_2) with coefficients in $\mathbb{Z}[x_{k+1}, \dots, x_n]$.
$x_3, \dots, x_7$

Remark following Timo's question:
In the context where a quiver encodes the data of an $\mathcal{N}=1$ 4d gaugetheory, the non-frozen nodes correspond to gaugegroups, the frozen ones to flavour symmetries.

4. What are Grassmanians ?

natural generalizations of projective spaces, and share many of their properties

We work over the field $\mathbb{C}$

4.11 Definition: The Grassmanian $Gr(k,n)$ is the set of all k-dimensional vector subspaces of $\mathbb{C}^n$.

ex: $Gr(1, n+1) \cong \mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$

$(x_0, \ldots\ldots, x_n) \sim (y_0, \ldots, y_n)$

if $\exists \ \lambda \neq 0$ s.t. $(x_0, \ldots, x_n) = (\lambda y_0, \ldots, \lambda y_n)$

Grassmanians can be thought of as a set of linear subvarieties of a projective space. A linear subvariety of $\mathbb{P}^n$ is a closed subvariety defined by linear homogeneous polynomials. An m-dimensional linear subvariety of $\mathbb{P}^n$ is a projective subvariety determined by an $(m+1)$-dim vector subspace of the vector space $\mathbb{C}^{n+1}$.

In the following I want to motivate the following

- Grassmanians are complex manifolds. (charts?) $\rightarrow$
- Grassmanians are projective algebraic varieties

$$\boxed{\underline{Gr(k,n)}} \longrightarrow \underline{\underline{\mathbb{P}^{\boxed{d}}}} \qquad d = ?$$

+ vanishing locus of hom. polynomials

4.21 Motivational example: $Gr(2,5)$

2-planes in $\mathbb{C}^5$

Let $\Lambda \in Gr(2,5)$ be a two-dim vector subspace in $\mathbb{C}^5$.

Choose basis vectors $(\tilde{a}_{11} \dots \tilde{a}_{15})$

$(\tilde{a}_{21} \dots \tilde{a}_{25})$

matrix $\begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{15} \\ \tilde{a}_{21} & \cdots & \tilde{a}_{25} \end{bmatrix}$

has full rank

(a_{ij}) and $(\tilde{a}_{ij})$ span the same subspace

if $\exists$ $\underline{g} \in GL(k) = \{$ invertible $k \times k$ matrices $\}$

$$(a_{ij}) = \underline{g} \, (\tilde{a}_{ij})$$

w.l.o.g.

$$\begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{15} \\ \tilde{a}_{21} & \cdots & \tilde{a}_{25} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & a_{15} \\ \underline{0} & \underline{1} & a_{23} & a_{24} & a_{25} \end{bmatrix}$$

6 entries (coordinates)

$\Rightarrow$ we can identify

$$\boxed{Gr(k,n) = \{ k \times n \text{ matrices of rank } k \} / GL(k)}$$

Consider

$$\begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{15} \\ \tilde{a}_{21} & \cdots & \tilde{a}_{25} \end{bmatrix} \longmapsto \left[\Delta_{(1,2)} : \cdots \; \Delta_{(i_1,i_2)} : \cdots : \Delta_{(4,5)} \right]$$

where $\boxed{\Delta(i_1,i_2)}$ denotes the 2x2 subdeterminant of $(\tilde{a}_{ij})$ formed by the columns $1 \le i_1 < i_2 \le 5$

this gives a well defined map

$$Gr(2,5) \longrightarrow \underline{\mathbb{P}^d} \qquad d = \binom{5}{2} - 1$$

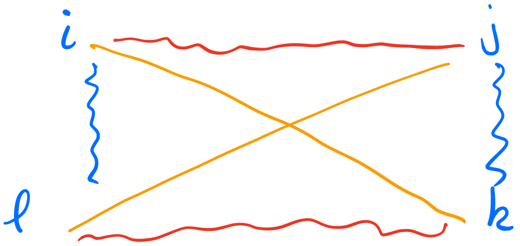
$$= \frac{5!}{3!\,2!} - 1 = 9$$

homogeneous polynomials ?

Although we expect $Gr(2,5)$ to be of dim 6, it embeds into $\mathbb{P}^9$, so there have to be equations satisfied...

for any $1 \le i < j < k < l \le 5$

$$\boxed{\underline{\Delta_{ik}\,\Delta_{jl}} = \underline{\Delta_{ij}\,\Delta_{kl}} + \underline{\Delta_{il}\,\Delta_{jk}}}$$



<u>ex</u>: verify

4.3 | Theorem

The Gassmanian $Gr(k,n)$ can be embedded as a complex submanifold of $\boxed{\mathbb{P}^{\binom{n}{k}-1}}$

Proof: Let $\underline{\Lambda \in Gr(k,n)}$ be a k-dim vector subspace in $\mathbb{C}^n$. Choose basis vectors $(a_{j1}, \dots, a_{jn})$, $\underline{j = 1, \dots, k}$ for Λ and form the row matrix of basis vectors

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \qquad k \times n \text{ matrix}$$

This matrix has full rank, since its rows are linearly indep. . Two matrices of full rank (a_{ij}) and (b_{ij}) span the same subspace iff there exists a matrix $\underline{g \in GL(k)} = \{\text{inv. } k\times k \text{ matrices}\}$ satisfying $(a_{ij}) = g\,(b_{ij})$

We can therefore identify

$$Gr(k, n) \cong G = \{ k \times n \text{ matrices of rank } k \} / \text{action of } GL(k)$$

Denote by $\boxed{\Delta_{(i_1 \dots i_k)}}$ the $k \times k$ subdet. of (a_{ij}) formed by the columns

$$1 \le i_1 < \dots < i_k \le n$$

$\langle 12 \rangle$

(Notation also $\langle i_1 \dots i_k \rangle$)

The mapping

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \longmapsto \left[\Delta_{(1, \dots, k)} : \dots : \Delta_{(i_1, \dots, i_k)} : \dots : \Delta(n-k+1, \dots, n) \right]$$

is well defined on the factor set $Gr(k, n)$

So we have a well defined map

$$Gr(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$$

it is injective and known as the Plücker embedding.

Consider a subspace Λ, where (a_{ij}) satisfies

$\boxed{\Delta(1,..,k) \neq 0}$. This kind of subspace Λ corresponds

to a unique matrix of the form

$$k\begin{pmatrix} \underbrace{\begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix}}_{k} & \overbrace{\begin{matrix} a_{1\,k+1} & \text{—} & a_{1n} \\ & & \vdots \\ a_{k\,k+1} & \cdots & a_{kn}\end{matrix}}^{n-k}\end{pmatrix}k$$

and each matrix of this form determines a unique subspace $\Lambda \in Gr(k,n)$. So there is a

bijective map

$$U_{(1..k)} = \{ \Lambda \in Gr(k,n) \mid \Delta_{(1..k)} \neq 0 \} \longrightarrow \boxed{\mathbb{C}^{k(n-k)}}$$

Because the open sets $\underline{U_{(i_1 \dots i_k)}}$ where

$\Delta(i_1 \dots i_k) \neq 0$ cover $Gr(k,n)$, these mappings

form an atlas of $Gr(k,n)$.

(Chart changes are given by multiplication by the rational fcts. $\boxed{\Delta_{I} / \Delta_{J}}$)

hence chart changes are analytic.

Also, because the Plücher embedding is given by holomorphic (rational) maps on the coord. charts, we know that $Gr(k,n)$ can be described as a complex submanifold of proj. space.

theorem $Gr(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ is a proj. alg. variety

proof see a book i.e. $\boxed{\text{Griffiths \& Harris}}$

5. Quiver and cluster algebra structure of Grassmannians

(Gekhtman, Shapiro, Vainshtein)

(C. Vergu review)

Let $\Lambda \in Gr(k,n)$ be represented

by a matrix $$\underbrace{\left(\boxed{\mathbb{1}_k} \, , \boxed{Y} \right)}_{n}$$

where Y is a $k \times \underbrace{(n-k)}_{\ell}$ matrix

Now define a matrix $\boxed{F_{ij}}$ for $1 \leq i \leq k$, $1 \leq j \leq \ell = (n-k)$

which is the biggest square matrix which fits inside Y

and whose lower-left corner is at position (i,j)

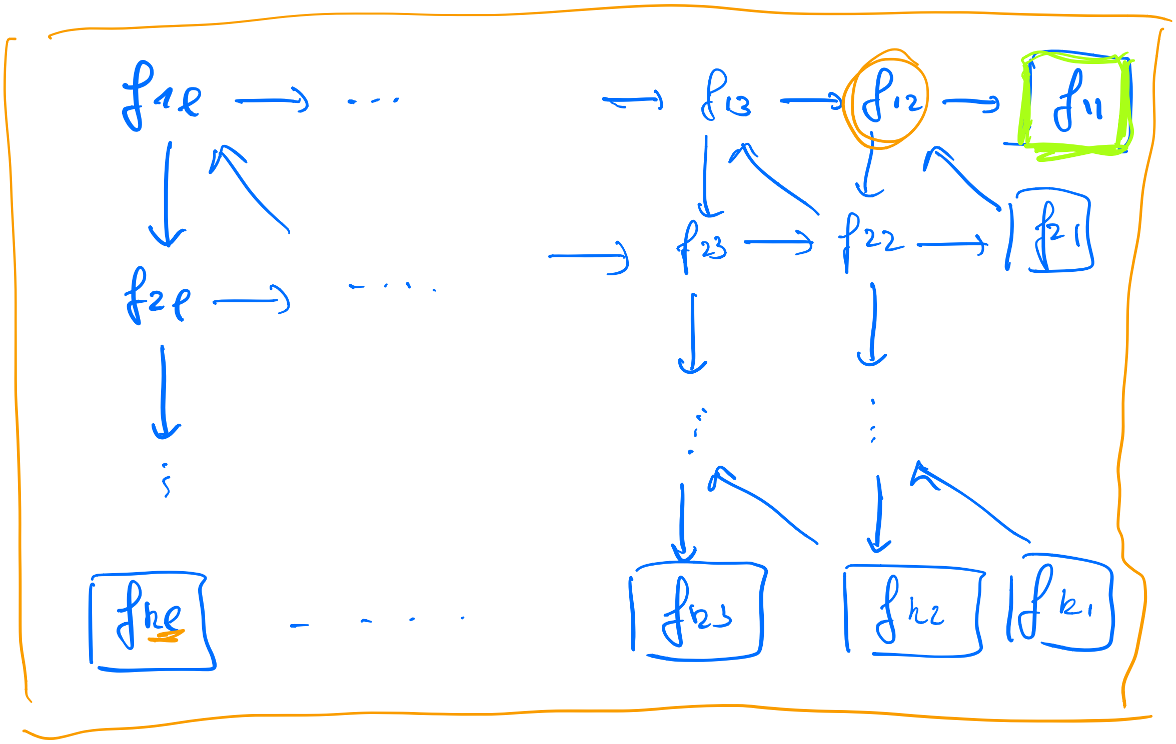
inside Y.

Then define $\ell(i,j) = \min(i-1, n-j-k)$ and

$$f_{ij} = (-1)^{(k-i)(\ell(i,j)-1)} \det F_{ij}$$

then the initial quiver for the $Gr(k,n)$ cluster

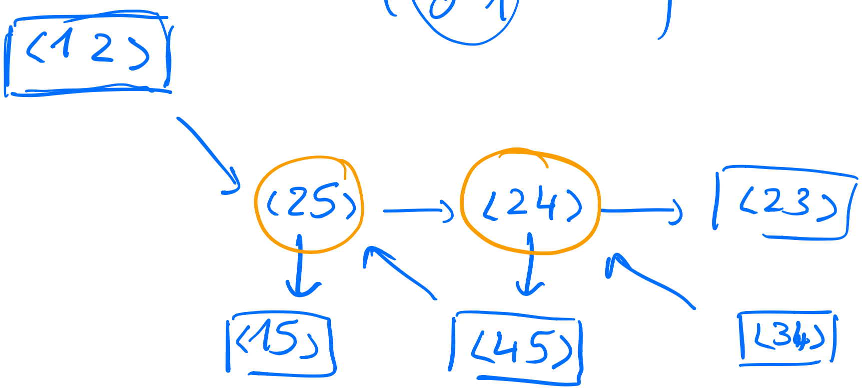
algebra is given by



The boxed variables are frozen vertices.

Ex.: $Gr(2,5)$ in terms of $\boxed{\langle ij\rangle = \Delta_{ij}}$ Plücker coordinates

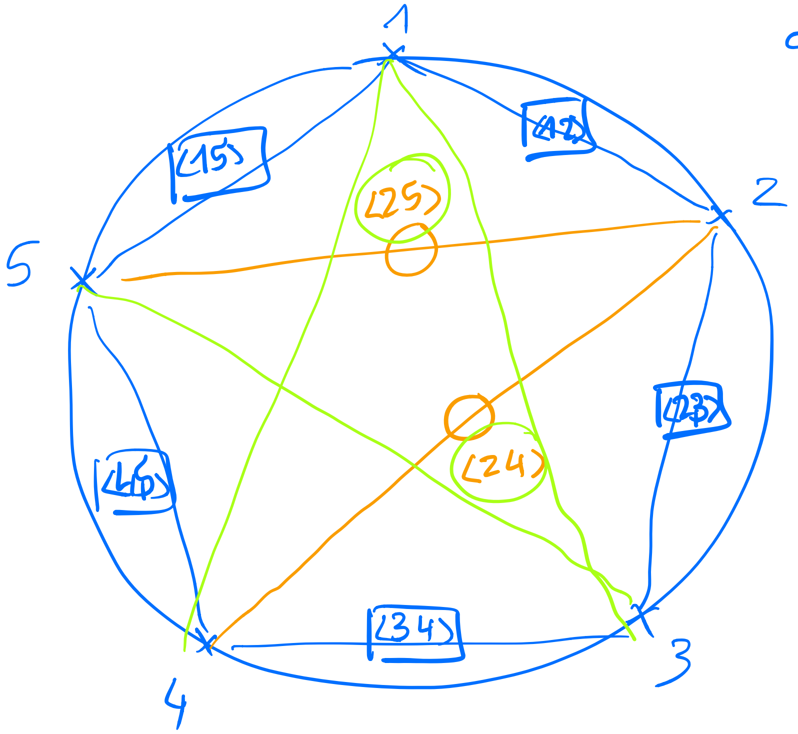
$$\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \end{pmatrix}$$



Rem: there is a bijection

$Gr(2,d)$

triangulation
of d-gon

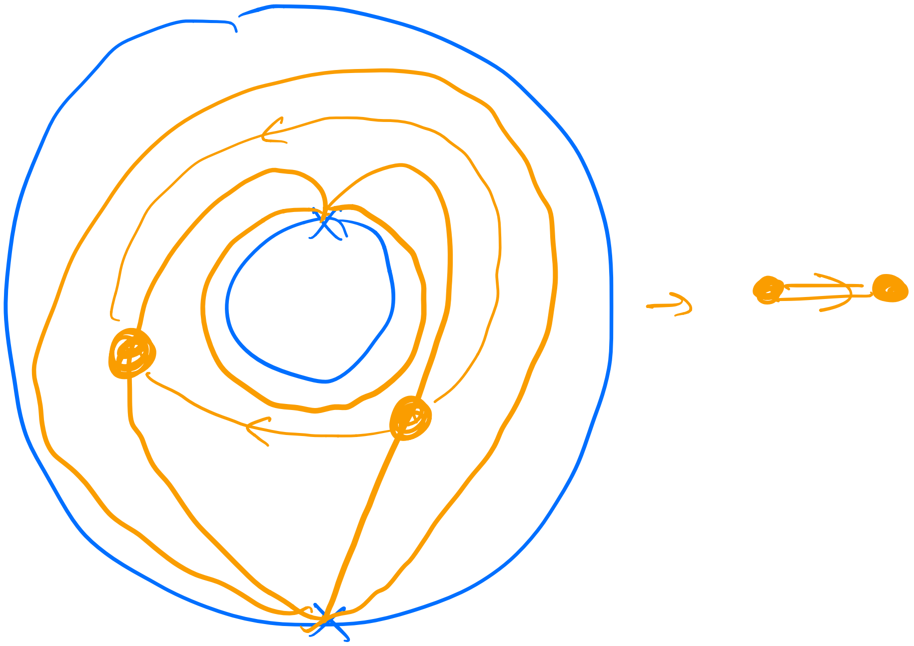


Remark following a question by Ingo

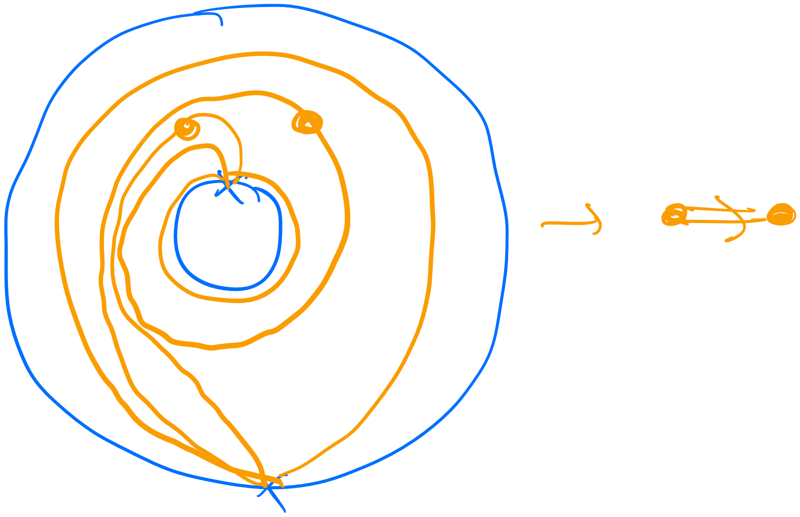
* there are quivers corresponding to triangulations which are of finite mutation type (topologically), whose cluster variables are not finite however, correspondingly

there are infinitely many triangulations.

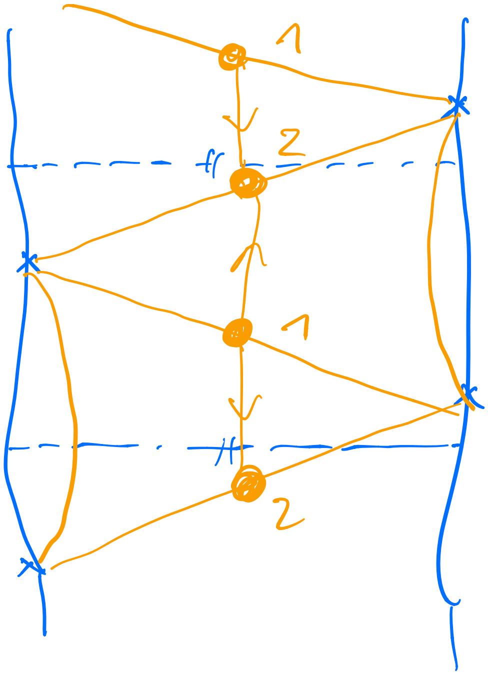
For example the cylinder with one marked point at each boundary



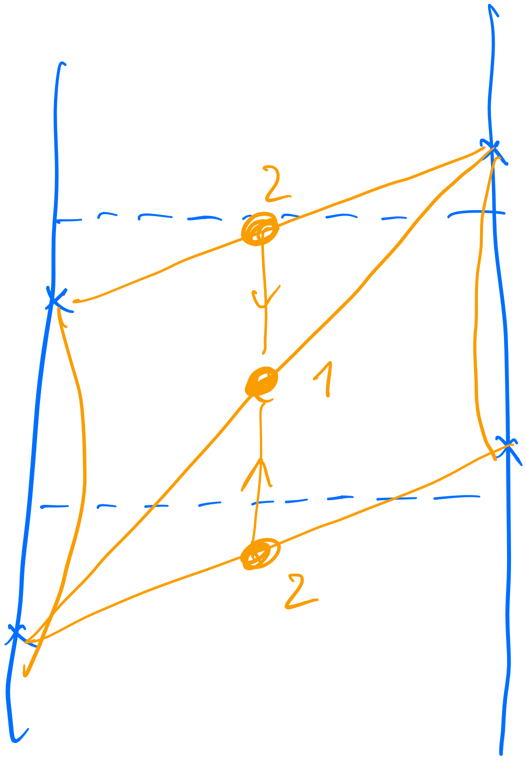
but also



easier to see:



mutate on 1



(added to the notes after the seminar)

Remark: This triangulation corresponds to the 2-Kronecker quiver

- its representation theory captures the BPS spectrum of Seiberg-Witten SU(2) theory

- In the context of last term's ZMP seminar it captures the coherent sheaves on $\mathbb{P}^1$, its mirror side captures geodesics of a quadratic differential on the SW curve

$$\Sigma = \left\{ y^2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right\}$$

$$\phi = \lambda^2, \qquad \lambda = y\,dz$$