2-groups

1. Motivation

What are groups?
⇝ Symmetries of an object in a category $\mathcal{C}$
$X \in \mathcal{C} \rightsquigarrow \mathrm{Aut}(X)$ is a group.

What are 2-groups?
⇝ Symmetries of an object in a 2-category + higher symmetries
$X \in \mathcal{C} \rightsquigarrow \mathrm{Aut}(X)$ is a 2-group

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<u>Example</u>: Consider a group H and its group of automorphisms.
How do H and $\mathrm{Aut}\, H$ interact with one another?

- $\mathrm{Aut}\, H$ acts on H
- $\mathrm{Ad}: H \to \mathrm{Aut}\, H$ is a group homomorphism:
  $\mathrm{Ad}_h(k) = h \cdot k \cdot h^{-1}$
- For $\varphi \in \mathrm{Aut}\, H$, $h \in H$:
  $\varphi \circ \mathrm{Ad}_h = \mathrm{Ad}_{\varphi(h)} \circ \varphi$
- H acts on $\mathrm{Aut}\, H$ by left multiplying with Ad_h:

$$\begin{tikzcd} \varphi \arrow[r, bend left, "h"] & \mathrm{Ad}_h \circ \varphi \arrow[r, bend left, "k"] & \mathrm{Ad}_{k \cdot h} \circ \varphi \end{tikzcd} \qquad \begin{tikzcd} \varphi \arrow[r, bend left, "h"] & \mathrm{Ad}_h \circ \varphi \arrow[l, bend left, "h^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} \varphi \arrow[d, "h"'] \\ \mathrm{Ad}_h \circ \varphi \end{tikzcd} \circ \begin{tikzcd} \psi \arrow[d, "k"] \\ \mathrm{Ad}_k \circ \varphi \end{tikzcd} \quad \underset{\text{Compose}}{=\!=\!=} \quad \begin{tikzcd} \varphi \circ \psi \arrow[d, "h \cdot \varphi(k)"] \\ \mathrm{Ad}_h \circ \varphi \circ \mathrm{Ad}_k \circ \psi \end{tikzcd}$$

$$= \mathrm{Ad}_{h \cdot \varphi(k)} \circ \varphi \circ \psi$$

→ $H \rtimes Aut(H)$ defines a monoidal category
 – a 2-group in fact.

associator $a: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$

Def: A (weak) 2-group is a (weak(ly)) monoidal category, such that:

- Every object is (weakly) invertible (w.r.t. $\otimes$)
- Every morphism is invertible.

$H \rtimes Aut H$ is an example of a more general construction:

Def: A crossed module of groups consists of

- Groups $(G, \circ), (H, \cdot)$ $G \to$ objects, $H \to$ morphisms
- an action $\alpha: G \to Aut\, H$ of G on H (denoted $g \triangleright h = \alpha_g(h)$)
- a group homomorphism $t: H \to G$ such that
  $t(h) \triangleright k = h \cdot k \cdot h^{-1}$
- Equivariance: $g \circ t(h) = t(g \triangleright h) \circ g$

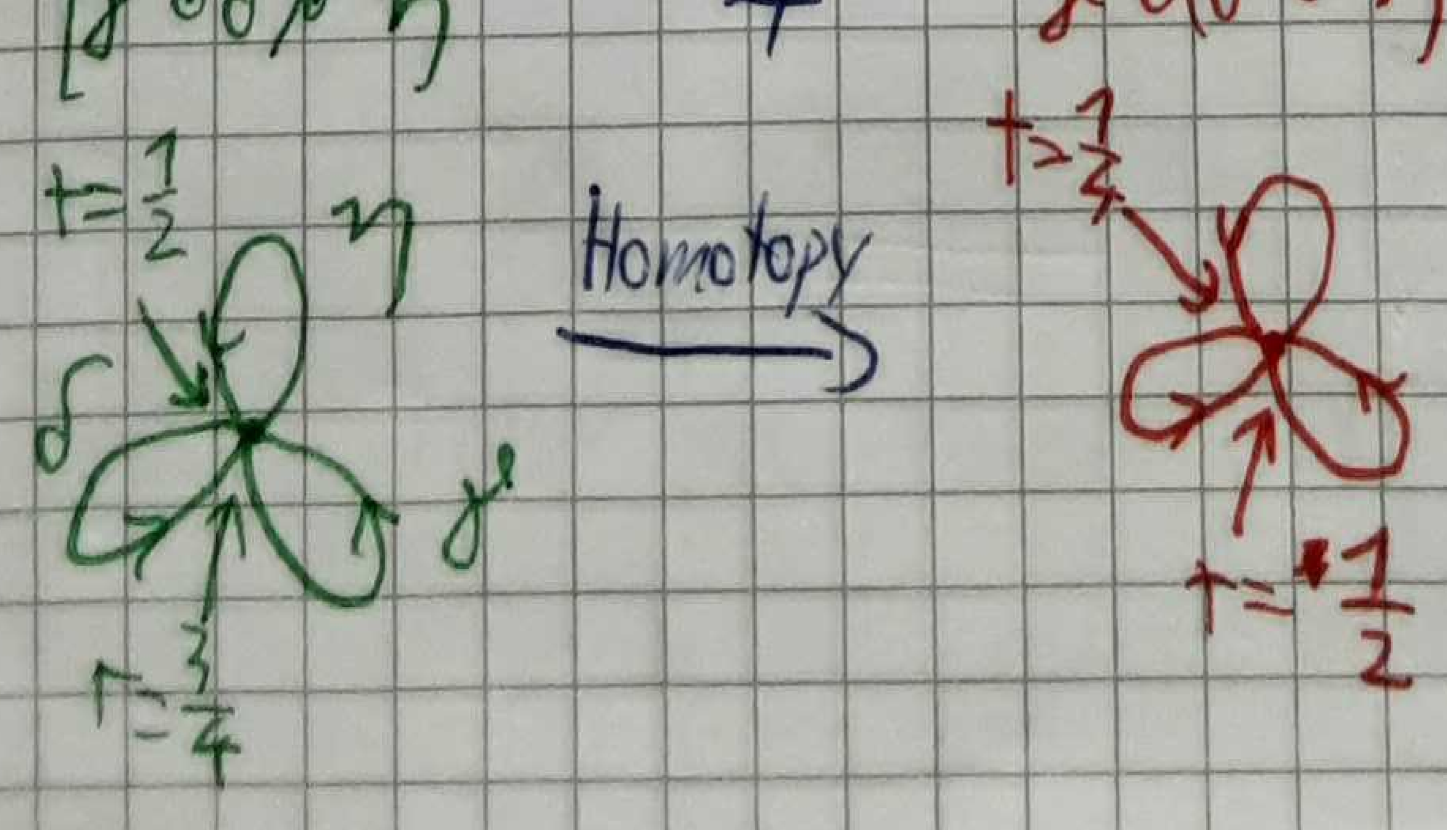
$\circ \rightsquigarrow$ ~~...~~ tensor product of objects, $\cdot$: composition of morphisms

$\alpha \rightsquigarrow$ tensor product of morphisms

Example: (Non-strict 2-group): The fundamental 2-group $\Pi_2(X, x)$ of a pointed topological space (X, x):

- Objects: loops $[0,1] \to X$ in x.
- Tensor product: Composition of loops
- Morphisms: Homotopies of loops (up to homotopy of homotopies)

$(\gamma \circ \delta) \circ \eta \neq \gamma \circ (\delta \circ \eta)$



Classification of 2-groups

Example $H \rtimes \mathrm{Aut}\, H$:

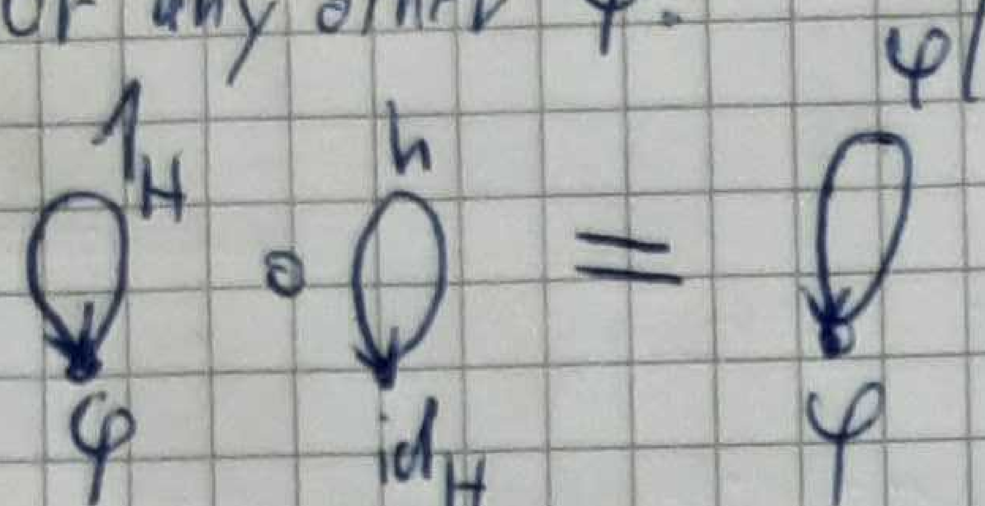
What are the automorphisms of $\varphi \in \mathrm{Aut}\, H$?

For $\varphi = id_H$:

$\rightsquigarrow h : id_H \to Ad_h \circ id_H = id_H$

$\iff h \in Z(H)$

For any other φ:

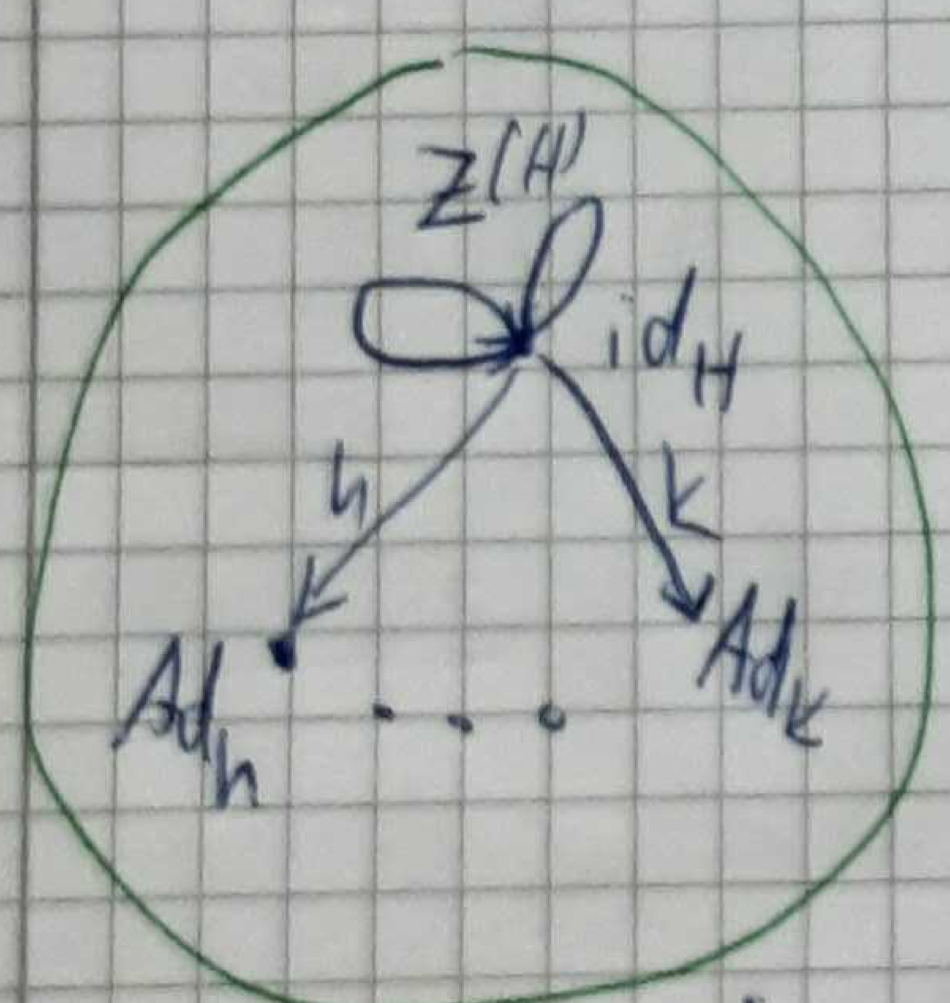


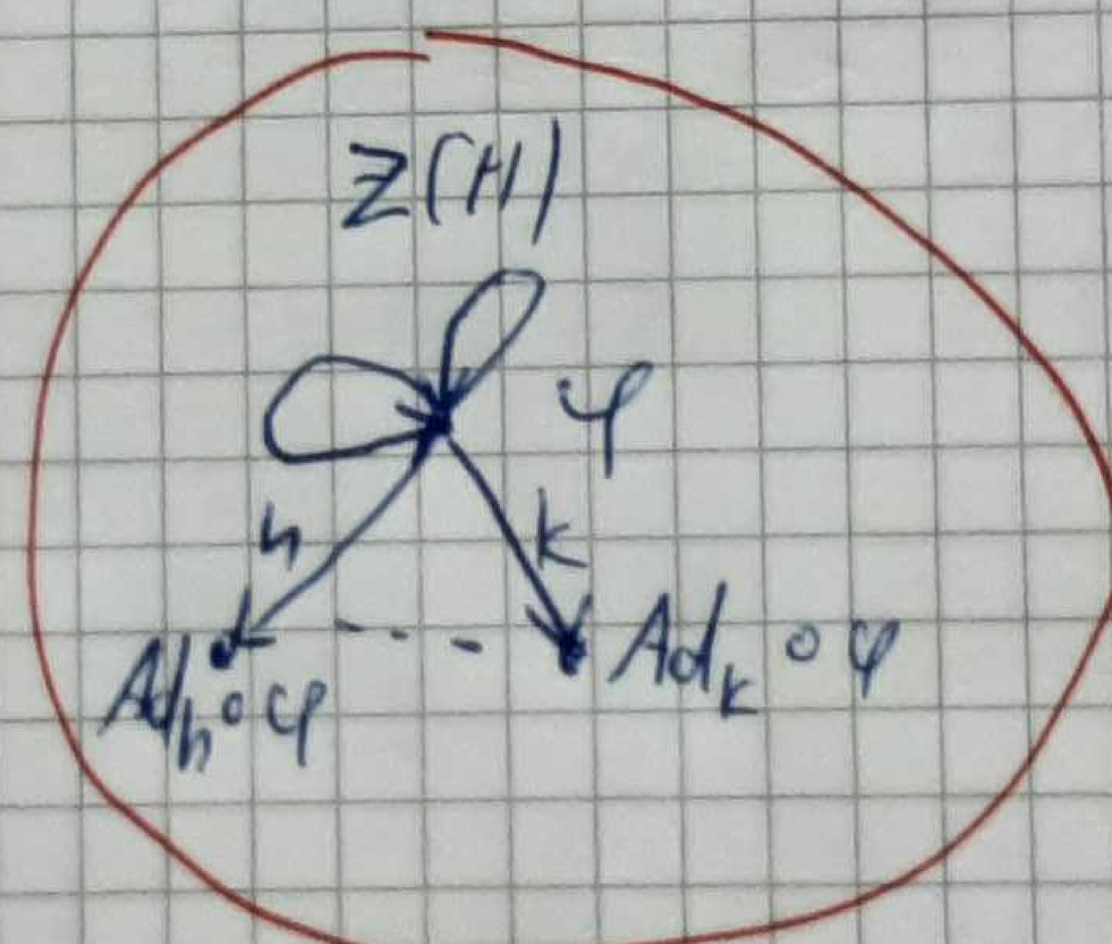
$\rightsquigarrow \varphi(h) \in \mathrm{Aut}(\varphi)$ for $h \in Z(H)$

$\to$ obtain $\mathrm{Aut}\, \varphi \cong Z(H)$

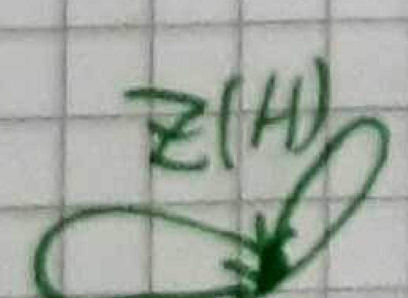
General 2-group: $\mathrm{Aut}(\varphi)$ is an abelian group.

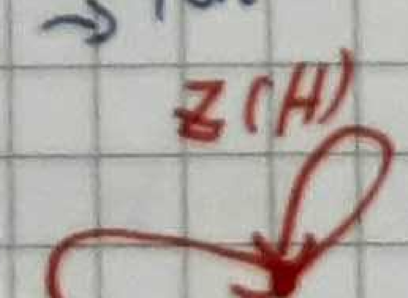
Skeleton of $\mathrm{Aut}\, H$:





$\downarrow$ Skeleton $\rightarrow$ identify isomorphic objects.





$\rightsquigarrow$ Skeleton: Objects are in $\mathrm{Aut}(G)/\mathrm{Inn}(G) = \mathrm{Out}(G)$ with $Z(G)$ as automorphisms of every object

(Compare picture w. topological group $G \rightsquigarrow G/G_0 = \pi_0(G)$)

Now let $H = \mathbb{Z}/4\mathbb{Z} \rightsquigarrow \mathrm{Aut}\, H \cong \mathbb{Z}/2\mathbb{Z} = \{id_H, \varphi\}$

The associator $a: \underset{\varphi}{\underset{\cong}{(\varphi \otimes \varphi)}} \otimes \varphi \to \varphi \otimes \underset{\cong}{(\varphi \otimes \varphi)}$ is trivial,

but we can twist a:

$a': \varphi \to \varphi$ can be any element $a' \in \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \in \mathbb{Z}/4\mathbb{Z}$.

$\rightsquigarrow$ obtain three non-equivalent monoidal structures, $\bar{1}, \bar{3}$ give equivalent ones.

<u>Theorem</u>: Classification of 2-groups)

Every 2-group is equivalent (as monoidal category) to a skeletal 2-group given by $(G, (M, \alpha), [\omega])$, where

- G a group $\rightsquigarrow$ Objects, $\otimes$ of objects $\hat{=}\ \mathrm{Out}(H)$
- M abelian group $\rightsquigarrow \mathrm{Aut}(1_G)$ $\hat{=}\ Z(H)$
- $\alpha: G \to \mathrm{Aut}(H)$ action $\rightsquigarrow$ $\otimes$ of morphisms $\hat{=}\ \mathrm{Out}(H) \circlearrowleft Z(H)$
- $[\omega] \in H^3(G, M) \rightsquigarrow$ associator $\hat{=}\ \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}/_{\bar{1} = \bar{3}}$

What is $[\omega] \in H^3(G, M)$?

ω: Normalized 3-cocycle, i.e. $\omega: G^{\times 3} \to M$ with

- $\omega(a, b, c) = 0$ if $a = 1, b = 1$, or $c = 1$ $\rightsquigarrow$ Compatibility with tensor unit
- $0 = a \triangleright \omega(b, c, d) - \omega(ab, c, d) + \omega(a, bc, d) - \omega(a, b, cd) + \omega(a, b, c) =: d\,\omega(a, b, c, d)$

  $\rightsquigarrow$ Pentagon identity for $a \otimes b \otimes c \otimes d$
- $[\omega]$: up to equivalence / im d

$$C^2(G, M) \xrightarrow{d} C^3(G, M) \xrightarrow{d} C^4(G, M)$$

$\rightsquigarrow H^3(G, M) = {}^{\ker d}/_{\mathrm{im}\, d} \ni [\omega]$

$$\begin{tikzcd} & \text{2-groups} & \\ \text{crossed modules} \arrow[ur, "\text{are}"] \arrow[rr, bend right, "\text{skeleton}"'] & & (G,(M,\alpha),[\omega]) \arrow[ul, "\text{classify}"'] \end{tikzcd}$$

<u>Further examples</u>

- Lie-crossed modules of Lie-group G and its Lie algebra $\mathfrak{g}$ $\rightsquigarrow \mathfrak{g} \rtimes G \cong TG$ (tangent space)

- $\underset{g\in G}{\overset{x\in\mathfrak{g}}{\circlearrowleft}} \cdot \underset{h\in G}{\overset{y}{\circlearrowleft}} = \underset{g\cdot h}{\overset{x+Ad_g(y)}{\circlearrowleft}}$

- Topological groups G + paths in G/homotopy
  - Objects $g,h\in G$
  - Morphisms: paths $g\to h$ up to homotopy

  e.g. $G=SO(3) \rightsquigarrow Aut(1_G)=\pi_1(SO_3)\cong \mathbb{Z}/2\mathbb{Z}$

  $\rightsquigarrow$ higher symmetries are $\mathbb{Z}/2\mathbb{Z}$,

  continuous rotation gives $\mathbb{Z}/2\mathbb{Z}$-symmetry

- for $G=\mathbb{Z}/n\mathbb{Z}=\{1,g,\dots,g^{n-1}\}$:

  $$H^3(\mathbb{Z}/n\mathbb{Z},M)\cong \{m\in M \mid (1+g+\dots+g^{n-1})\triangleright m=0\}/(1-g)\triangleright M$$

  for instance for $\mathbb{Z}/2\mathbb{Z}$ and $g\triangleright m=-m$:

  $$(1+g)\triangleright m = m+g\triangleright m = m-m=0$$

  $$\rightsquigarrow H^3(\mathbb{Z}/2\mathbb{Z},M)=M/_{m=-m}$$

  for $M=\mathbb{Z}/4\mathbb{Z}$:

  $$H^3(\mathbb{Z}/2\mathbb{Z},M)=\{\bar{0},\bar{1},\bar{2},\bar{3}\}/_{\bar{1}=\bar{3}}$$