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Kramers-Wannier duality defects in 3+1 dimensions

Plan

(i) Introduction / Recap of Ising 2d
(ii) Higher form symmetries, gauging & anomalies
(iii) KW duality defects. & 2 construction $\begin{cases} \text{Half gauging} \\ \text{'t Hooft anomalies} \end{cases}$
(iv) Example(s)

(i) We saw that the topological defect line of Ising CFT, $\mathcal{D}_{M(3,4)}$ form a category (Tambara-Yamagami)

$\mathbb{1}, \varepsilon, \mathcal{N}$ $\quad \{\mathbb{1}, \varepsilon\} \cong \mathbb{Z}_2 \quad \mathcal{N} \times \mathcal{N} = \mathbb{1} + \varepsilon$

$\varepsilon \times \mathcal{N} = \mathcal{N}$

In general a TY category is a (fusion) category:

$(g_1, \dots, g_n) \cong G^{(n)}$ $\quad g_i \times \mathcal{N} = \mathcal{N} \quad \mathcal{N} \times \mathcal{N} = \sum_{i=1}^{n} g_i$

$\mathcal{N} \times g_i = \mathcal{N}$

This is actually the "simplest" category being the extension of a group G by a simple (non-invertible) element
We've also learnt that the (non critical) ising model enjoys a non trivial high-low temperature <u>duality</u> $Z(\beta) \propto Z(\tilde\beta)$

$\tilde\beta = -\frac{1}{2} \ln \tanh \beta$.

At the critical point where $\tilde\beta = \beta$, this is a symmetry of the Ising CFT: the CFT is self dual under Kramers-Wannier.

Specifically $T = \text{Ising}_{CFT}$ $\quad T \cong T/\mathbb{Z}_2$: the theory is self dual under the gauging of the $\mathbb{Z}_2$ symmetry generated by η

How can we understand this statement?
Gauging a continuous symmetry in a QFT, accounts for introducing a background connection for the symmetry and promoting this to be a dynamical field of the theory (i.e. path. integrating over it in the partition function)

T is invariant under G $\rightarrow$ $Z[\{A_i\}, B] = \int \mathcal{D}\phi \, e^{-iS[\phi; \{A_i\}, B]}$

$A_i^{(1)} \in \Omega^1(M, G_i)$ $\quad B^{(1)} \in \Omega^1(M, H)$

Gauging H : $Z[\{A_i\}] = \int \mathcal{D}\phi \int \mathcal{D}_{B^{(1)}} \, e^{-iS[\phi, B^{(1)}; \{A_i\}]}$

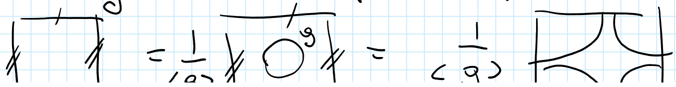
When H is discrete : $B^{(1)} \in H^1(M, H)$

$Z[A_i, B] = \int \mathcal{D}\phi \, e^{-iS[\phi, B, \{A_i\}]}$

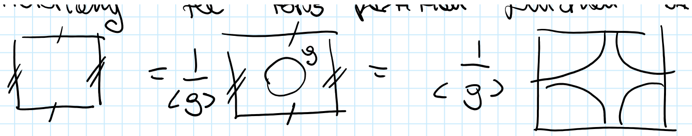
and gauging H means : $Z[\{A_i\}] = \sum_{B^{(1)} \in H^1(M,H)} \int \mathcal{D}\phi \, e^{-iS[\phi, B^{(1)}; \{A_i\}]}$

i.e. I insert networks of symmetry defects in all the possible cocycles

Pictorially the torus partition function of a 2d theory is:



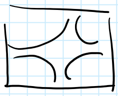
[illegible] the torus partition function [illegible]



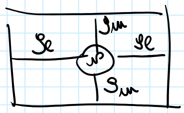
g is a defect in the TDCategory of the theory

if TDC is TY, then take $g = \mathcal{N}$.

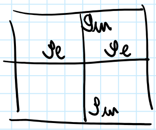
$$\mathcal{Z}(T) = \frac{1}{\langle \mathcal{N} \rangle}$$



$$= \frac{1}{\langle \mathcal{N} \rangle} \sum_{\rho, m = 1}^{n}$$



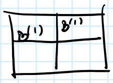
$$= \sum_{\rho, m = 1}^{n}$$



$$\mathcal{N} \times \mathcal{N} = \sum g_i$$

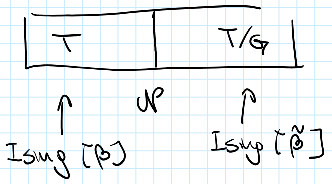
$$= \sum_{B^{(1)} \in H^1(T^2, G)} \mathcal{Z}(T) = \mathcal{Z}[T/G]$$

For Ising $\quad \mathcal{Z}(T) = \sum_{B^{(1)} = \{1, \eta\}}$



$= \mathcal{Z}(T/\mathbb{Z}_2)$

More generally $\mathcal{N}$ can be thought as an interface (dualities defect) between the two theories T and T/G



when $T \cong T/G$ then $\mathcal{N}$ is a topological (symmetry) defect and the symmetry structure of T no longer forms a group but rather a fusion category (TY) w/ non invertible elements.

(ii)

Now, we want to find an higher dimensional analogues of this structure. In order to do this we will need 2 notions:

1. higher-form symmetry & gauging
2. 't Hooft Anomalies

Let's start w/ the first

Physicist already know from SC what h.f.s. are

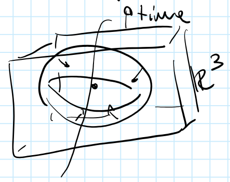
For mathematicians/recap

	0-form symmetry	p-form symmetry
Charged obj	local op dim 0	extended ops [dim p]
Charge ops	$\mathrm{cod}(1) = d-1$ topological operators $U_g(M^{d-1})$	$\mathrm{cod}(p+1) = d-p-1$ topological operators $U_g(M^{d-1-p})$
Back connection	$A^{(1)} \in \Omega^{(1)}(M, \tilde{G})$ cont. $\in H^1(M, \tilde{G})$ discrete flat (discrete) connection $\to$	$A^{(p+1)} \in \Omega^{(p+1)}(M, \tilde{G})$ $A^{(p+1)} \in H^{p+1}(M, \tilde{G})$
Linking rule	$U_g(M^{(d-1-p)})\, V_\rho(\Gamma^{(p)}) = R_\rho(g)\, V_\rho(\Gamma^{(p)})$ linking number in spatial slice (Alexander duality) $\int_{M_d} \langle A^{(p+1)}, B^p \rangle$	

$\tilde{G} = \mathrm{Hom}(G, U(1)) \cong G$
whenever G is finite
or $U(1)$ so $\tilde{G} \cong G$
and forget about it.

Today: 1-form symmetry in 4 dim

charged ops: lines (not-topological)
charge (top) operators: surfaces dim 2



Fact: H.F.S.^{p} are abelian on any space
with $H^{(p)}(M) \simeq \{e\}$ (

Gauging a 1-form symmetry $G^{(1)}$

$$Z[A^{(2)}] = \sum_{A^{(2)} \in H^2(M, G)} e^{iS[\phi; A^{(2)}]}$$

Fact: if T has a q-form symmetry $G^{(q)}$
$(T/G^{(q)})$ has a $d-q-2$ form symmetry
Intuitively: $S[\phi, A^{(2)}] \supset \int \langle A^{(q+1)}, \cup B^{(d-q-1)} \rangle$
e.om. $\Rightarrow B \in H^{d-q-1}(M, \tilde{G})$
$\tilde{G} = \mathrm{Hom}(G, U(1))$ (Pontryagin dual group)

$\Rightarrow [T/G^{(q)}]$ has a $\tilde{G}^{(d-q-2)}$ dual symmetry.

But then if I want KW symmetry:

$T \cong T/G^{(q)}$ all the symmetries have to
match in order to be a symmetry

$\Rightarrow q = d-2-q \Rightarrow \boxed{q = \frac{d-2}{2}}$

So in 4d I must gauge a 1-form symmetry.
Notice: only in even # of dim! for d odd?
[Maybe I have to gauge a fermionic symmetry!]
(in $d=3$ gauging susy).

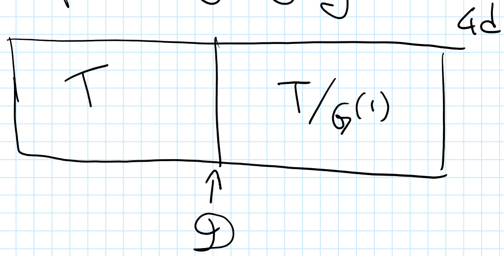
(iii) Take $d=4$ T with $\Gamma^{(0)}$ and $\Gamma^{(1)}$
We further assume that both $\Gamma^{(0)}$ and $\Gamma^{(1)}$ are anomaly-free.
So they can both be gauged.
Final result (we will get there):
If I gauge $\Gamma^{(0)}$ $T/\Gamma^{(0)}$ has an higher group symmetry
$T/\Gamma^{(1)}$ has a TY symmetry

Probably I don't have time

A la! Show: T has a $G^{(1)}$ w/ charged ops $V(\Gamma^{(1)})$ and symmetry ops $U(\Sigma^{(2)})$ g

p

half space gauging



When I gauge, I promote $A^{(2)}$ to be dynamical
$A^{(2)} \to a^{(2)}$

$da^{(2)} = 0$ as flat connection $\Rightarrow \mathcal{D}$ is topological interface

as $\mathcal{D} - \mathcal{D}' = \int_V da = 0$

$T/G^{(1)}$ has a dual $(4-2-1=)$ 1-form symmetry $\tilde{G}^{(1)}$
w/ topological ops $\exp\left(i g \oint_{\Sigma^{(2)}} a^{(2)}\right) = \eta_g(\Sigma^{(2)})$

So, when T and $T/G^{(1)}$ are not equivalent (duality)
then I have $G^{(1)}_{\text{of } T}$ generated by $U_g(\Sigma^{(2)})$ and $\tilde{G}^{(1)}$ of $(T/G^{(1)})$
generated by $\eta_g(\Sigma^{(2)})$

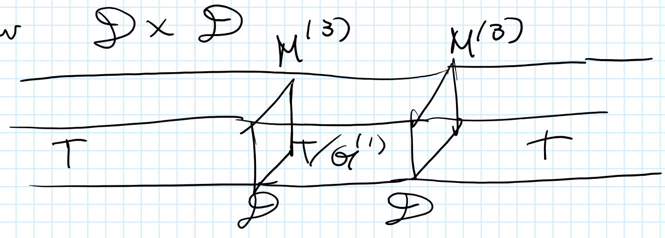
If $T \cong T/G^{(1)}$ then $\mathcal{D}^{(3d)}$ is not an interface (domain wall)
but a defect of the theory

so the theory has a bigger symmetry structure as there are
top ops of $G^{(1)}$ $U_g(\Sigma^{(2)})$ and $\mathcal{D}(M^{(3)})$

Let's analyse it thoroughly:

since $a^{(2)}\big|_{\mathcal{D}} = 0$ $\quad \eta_i \times \mathcal{D} = \mathcal{D} \times \eta_i = \mathcal{D}$ as η_i are annihilated on $\mathcal{D}$

Now $\mathcal{D} \times \mathcal{D}$



$\Rightarrow$ Inserting two $\mathcal{D}$ accounts for gauging

$G^{(1)}$ only on $M^{(3)}\times I$

$$\Rightarrow \mathcal{D}\times\mathcal{D} = \sum_{S\in H^{(2)}(M^{(3)}\times I,\, \partial(M^3\times I),\, G^{(1)})} \eta(S) = \sum_{\Sigma\in H_2(M\times I,\, G^{(1)})} \eta(\Sigma^{(2)}) = \sum_{\Sigma\in H_2(M,\, G^{(1)})} \eta(\Sigma)$$

cocycle exact on the $\partial(M^3\times I)$

Lefschetz duality (Poincaré)

$= \sum \eta_i$ — Schematically

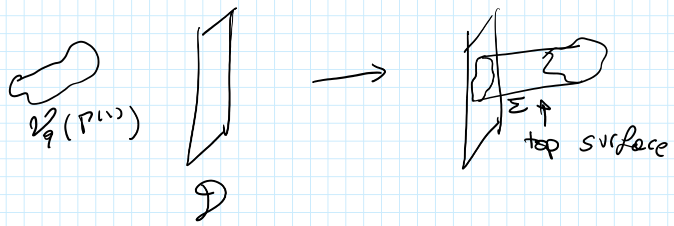
In general $\mathcal{D}\times\mathcal{D} = \#\sum \eta_i$ ← cod 2

cod 1

constant computable, e.g. for $G^{(1)}=\mathbb{Z}_N$ $\# = \frac{1}{N}$

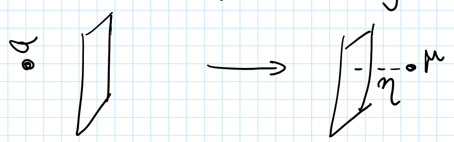
$\Rightarrow$ Symmetry category: $\{\eta_i\}\cong G^{(1)}$ $\quad \eta\times\mathcal{D} = \mathcal{D}$

TY category $\quad \mathcal{D}\times\mathcal{D} = \sum_i \eta_i$



$\partial\Sigma = \underset{\cap\,\mathcal{D}}{\cup} \cup \Gamma^{(1)}$

cfr. Ising



(IV) Example $\quad U(1)$ YM $\quad \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$

$U_e^{(1)}\times U_m^{(1)}$ $\qquad \frac{1}{g^2}\int F\wedge\star F + \frac{i\theta}{2\pi}\int F\wedge F$

T	$T/G^{(1)}$

When $\frac{a\tau+b}{c\tau+d}$

$T/\mathbb{Z}_N^{(1)}$: $A\to$ "A/N" ← $\mathbb{Z}_N$ connection

$\tau \qquad \frac{\tau}{N^2}$

when $\frac{\tau}{N^2} = \frac{a\tau+b}{c\tau+d}$ $\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$

$\Rightarrow T/\mathbb{Z}_N^{(1)}\cong T \qquad \tau = iN$ is as such

$$S = \frac{N}{4\pi}\int_{x<0} dA_L\wedge\star dA_L + \frac{N}{4\pi}\int_{x>0} dA_R\wedge\star dA_R + \frac{iN}{2\pi}\oint_{x=0} A_L\wedge dA_R$$

the $\mathcal{D} = \exp \frac{iN}{2\pi} \oint_{x=0} A_L \, dA_R$ properly q. C.S. term

(Why? Well e.o.m. $\Rightarrow dA_L \Big|_{x=0} = \frac{1}{N} d\tilde{A}_R \Big|_{x=0} = -i \, A \, dA_R \Big|_{x=0}$)

gauged

$\Rightarrow$ Imposes that $T_R = T_L / \mathbb{Z}_N^{(1)}$

$\boxed{\text{One can explicitly work out } \mathcal{D} \times \mathcal{D} = \sum M_i .}$