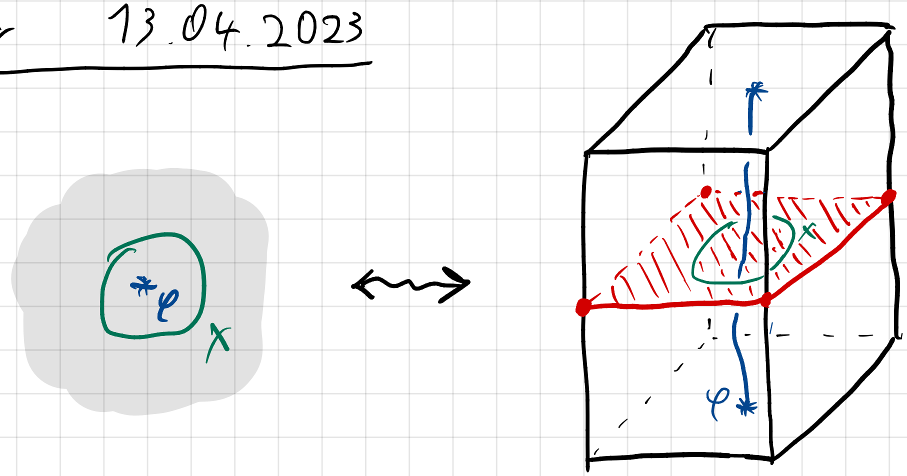
Topological defects in the 2d Ising CFT

ZMP-Seminar 13.04.2023

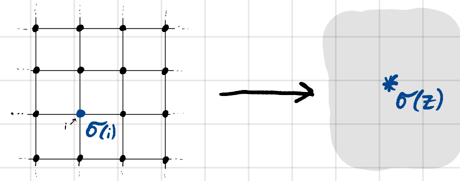


Outline:

1) 2d perspective
   - Ising CFT
   - Topological defects
2) 3d perspective
   - Ising MFC
   - 2d CFT from 3d TFT
   - Some correlators

2d perspective

Ising CFT



Taking a continuum limit of the lattice Ising model at T_{crit} leads to a $(4,3)$, $c=\frac{1}{2}$ Virasoro minimal model with state space
i.e space of states on circle

$$\boxed{\mathcal{H} = R_0 \otimes \bar{R}_0 \oplus R_{1/2} \otimes \bar{R}_{\frac{1}{2}} \oplus R_{\frac{1}{16}} \otimes \bar{R}_{\frac{1}{16}}}$$

R_h ... unitary highest weight rep with weight h

i.e we have three primary fields $\mathbb{1}$, ε, σ

$\mathbb{1}$: identity

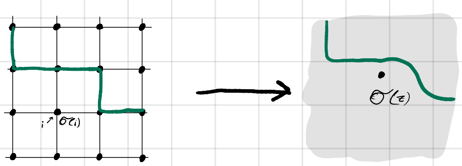
ε: energy (corresponds to perturbation in temperature)

σ: spin (corresponds to perturbation in magnetic field)

Topological defects

In the lattice description we saw that a local change along a path leads to a modification of the partion function

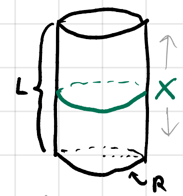
$\Rightarrow$ This should be reflected in the CFT



$\Rightarrow$ Defect operator!

Moreover this should be topological because only the endpoints were relevant!

Putting the system on a cylinder we get an operator $D_x : \mathcal{H} \to \mathcal{H}$



$e^{-\frac{L}{2}H(R)} D_x \, e^{-\frac{L}{2}H(R)} \xrightarrow{L\to 0} D_x$

(topological implies that we can move it up and down i.e. $[D_x, H] = 0$
↑ Hamiltonian)

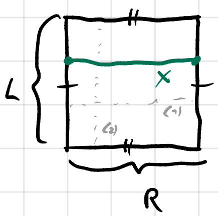
How do we constrain this operator?

i) Defect topological $\Leftrightarrow$ $[T, D_x] = [\bar{T}, D_x] = 0$ $\Leftrightarrow$ $[L_n, D_x] = [\bar{L}_n, D_x] = 0$ $\forall n \in \mathbb{Z}$ $\Rightarrow$ D_x is $\mathrm{Vir} \times \overline{\mathrm{Vir}}$ interdwiner

irreducibility of R_h implies that it needs to be of the form

$$D_x = \alpha P_0 + \beta P_{1/2} + \gamma P_{1/16} \quad \text{with } P_h : \mathcal{H} \to R_h \otimes \bar{R}_h \text{ projectors and } \alpha, \beta, \gamma \in \mathbb{C}.$$

ii) Let us calculate the torus amplitude



cut in two ways:

(1) $\quad tr_{\mathcal{H}}\left(D_x \, e^{-LH(R)}\right)$

(2) $\quad tr_{\mathcal{H}_x}\left(e^{-RH_x(L)}\right)$

↑

$\mathcal{H}_x = \bigoplus_{i,j} M_{i,j} R_i \otimes \bar{R}_j$, with $i,j \in \{0, \frac{1}{2}, \frac{1}{16}\}$ and $M_{i,j} \in \mathbb{N} = \mathbb{Z}_{\geq 0}$

(1)
$$
\begin{aligned}
tr_{\mathcal{H}}\left(D_x \, e^{-LH(R)}\right) &= tr_{\mathcal{H}}\left((\alpha P_0 + \beta P_{1/2} + \gamma P_{1/16})\, e^{-LH(R)}\right) \\
&= tr_{\mathcal{H}}\left((\alpha P_0 + \beta P_{1/2} + \gamma P_{1/16})\, q^{L_0 + \bar{L}_0 - \frac{c}{12}}\right) \quad \text{with } q = e^{-2\pi L/R} \\
&= \alpha \chi_0(q)\chi_{\bar{0}}(q) + \beta \chi_{\frac{1}{2}}(q)\chi_{\bar{\frac{1}{2}}}(q) + \gamma \chi_{\frac{1}{16}}(q)\chi_{\bar{\frac{1}{16}}}(q)
\end{aligned}
$$

↑ character for the rep R_0.

2) $\quad tr_{\mathcal{H}}\left(e^{-RH_x(L)}\right) = \ldots = \sum_{i,j} M_{i,j}\, \chi_i(\tilde{q})\chi_{\bar{j}}(\tilde{q}) \quad$ with $\tilde{q} = e^{-2\pi R/L}$

Using the modular transformation of characters under S-transformation $\left(\chi_i(q) = S_{ij}\, \chi_j(\tilde{q}) \right)$ given by the S-matrix leads to the linear system

$\chi_0(q) = \frac{1}{2}\left(\chi_0(\tilde{q}) + \chi_{\frac{1}{2}}(\tilde{q}) + \sqrt{2}\, \chi_{\frac{1}{16}}(\tilde{q})\right)$

$$\boxed{S M S^{-1} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}} \qquad \text{with} \qquad S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

and $M \in Mat_{3\times 3}(\mathbb{N})$.

We can find "minimal" solutions which form a basis for the D_X:

$$\boxed{\begin{aligned} D_{\dot{1}} &= P_0 + P_{1/2} + P_{1/16} = id_{\mathcal{H}} \\ D_{\dot{\varepsilon}} &= P_0 + P_{1/2} - P_{1/16} \\ D_{\dot{\sigma}} &= \sqrt{2}\, P_0 - \sqrt{2}\, P_{1/2} \end{aligned}}$$

"no defect"

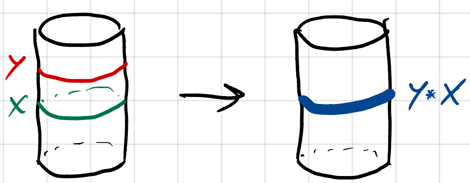
"spin flip"

"???"

(Labels will make sense soon)

What do they interact?

First note that we can fuse defects by moving them together:

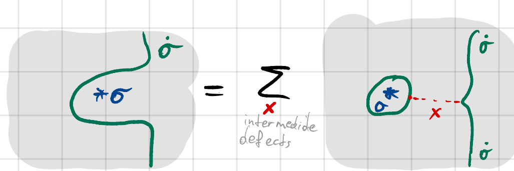


$\Longleftrightarrow \quad D_Y \circ D_X = D_{Y*X} \quad$ e.g. $\begin{cases} D_{\dot{\varepsilon}} \circ D_{\dot{\varepsilon}} = D_{\dot{1}} \\ D_{\dot{\varepsilon}} \circ D_{\dot{\sigma}} = D_{\dot{\sigma}} \circ D_{\dot{\varepsilon}} = D_{\dot{\sigma}} \\ D_{\dot{\sigma}} \circ D_{\dot{\sigma}} = D_{\dot{1}} + D_{\dot{\varepsilon}} \end{cases}$

$\{D_{\dot{1}}, D_{\dot{\varepsilon}}\} \cong \mathbb{Z}_2 \leftarrow$ Symmetry defect!

$D_{\dot{\sigma}}$ is non-invertible, but what is it?

We could pull a $\dot{\sigma}$ defect over a σ bulk field:



Use 3d perspective to gain insight!

3d perspective

Idea: Rational CFT's are boundary theories of a 3d TFT, let us use this to describe the defect operators:

Ising MFC

two possibilities: $\pm\sqrt{2}$

$\mathcal{I}_s$: 3 simples: $\underset{R_0}{\mathbb{1}}, \underset{R_{1/2}}{\varepsilon}, \underset{R_{1/16}}{\sigma}$ + data (associator, ...)

Rep(VOA)

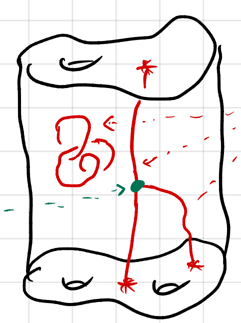
Fusion rules:

⊗	𝟙	ε	σ
𝟙	𝟙	ε	σ
ε	ε	𝟙	σ
σ	σ	σ	$\mathbb{1}\oplus\varepsilon$

fusion ⟺ $\mathbb{C}$-linear, semisimple finite, monoidal, rigid

One can show that this fusion cat can be equipped with a braiding (actually four distinct ones) which is non-degenerate, i.e. we get a modular fusion category. (two distinct pivotal structures)

Claim: There is a construction which produces a 3d TFT from a modular fusion category: (Reshethikin-Turaev)



Wilson* line operators labeled by objects in MFC

*actually ribbon

*framed

junction operator labeld with morphism

Aside: 3d TFT: $Z_{RT}: Bord_{3,2}(\mathcal{I}_s) \longrightarrow vect_{\mathbb{C}}$

$\Sigma \longmapsto Z(\Sigma)$ ... vector space $\hat{=}$ state space

$\Sigma \xrightarrow{M} \Sigma' \longmapsto Z(M): Z(\Sigma) \to Z(\Sigma')$ ... linear map $\hat{=}$ operators

compatible with $\otimes_{\mathbb{C}}$ ...

2d CFT from 3d TFT

CFT	TFT
chiral block space	state space
correlator	special vector

(Theorem in algebra, otherwise convergence problems $g>1$)

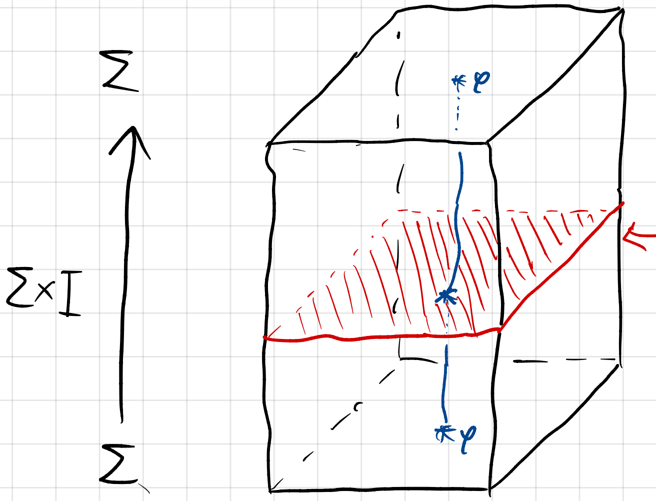
$$Bl^{\text{chiral}}(\Sigma_{g,n};(X_1,\dots,X_n)) \cong \mathcal{Z}_{RT}(\Sigma_{g,n};(X_1,\dots,X_n))$$

VOA-Reps

$$\cong \mathrm{Hom}_{\mathcal{I}_s}(1\!\!1, X_1\otimes\dots\otimes X_n\otimes L^{\otimes g})$$

$L=\bigoplus_{\text{simples}} X^*\otimes X$

Extra input datum to specify CFT



special surface defect

Apply $\mathcal{Z}$

$$Cor_\Sigma \in End(\mathcal{Z}(\Sigma)) \cong \mathcal{Z}(\Sigma)\otimes\mathcal{Z}(\Sigma)^*$$

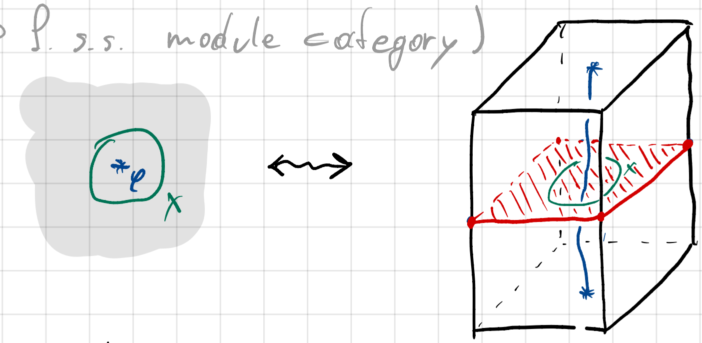
chiral   antichiral

For us $\boxed{}$ = triv

(in general it corresponds to a special symmetric Frobenius algebra $\Leftrightarrow$ f.s.s. module category)

What about defects?

A line defect in the CFT corresponds now to a line defect in the red plane.



Since the red plane is trivial for us, it is just a regular Wilson line and labeled by an object in $\mathcal{I}_s$!

(In general it would be a bimodule for the above algebra)

The action of a line defect can now be computed as follows:

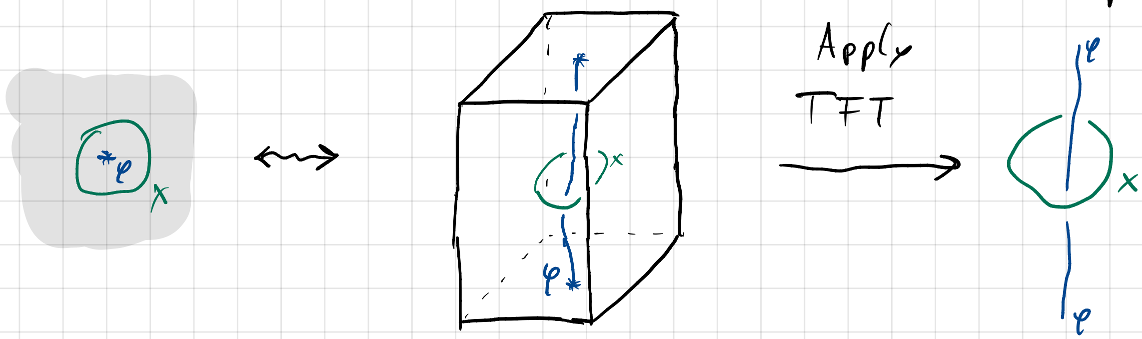


diagram in $\mathcal{I}_s$.

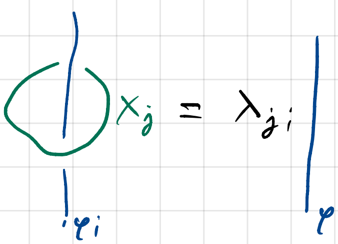
$\Updownarrow$

morphism

Since X, φ are both objects in $\mathcal{I}_s$ we can decompose them into simples

$\Rightarrow$ We only need to understand how simples act!

The action of a simple defect X_j on a simple field φ_i



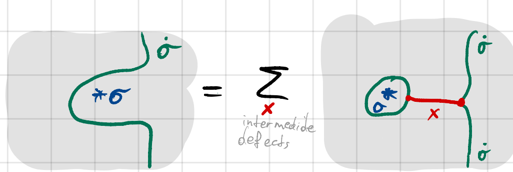
because φ_i is simple $\xrightarrow{\text{trace}}$ $\lambda_{ji} = \frac{s_{ji}}{d_i} = \frac{S_{ji}}{S_{11}\, d_i}$

that link in   s-matrix

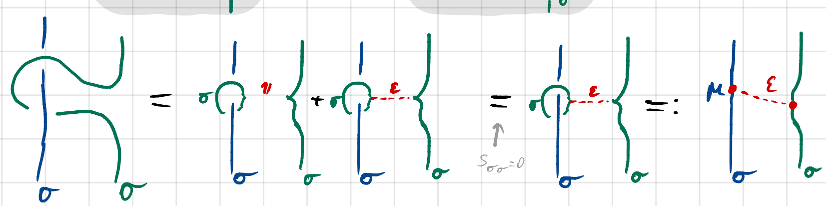
$\mathrm{End}(\varphi_i) \cong \mathbb{C}$

$$\Rightarrow \boxed{\begin{aligned} D_1 &= id_1 + id_\varepsilon + id_\sigma \\ D_\varepsilon &= id_1 + id_\varepsilon - id_\sigma \\ D_\sigma &= \sqrt{2}\, id_1 - \sqrt{2}\, id_\varepsilon \end{aligned}}$$

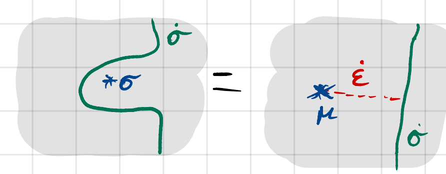
as before !!!

What about  ?

We get

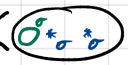
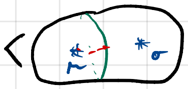
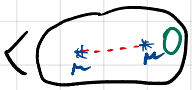
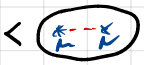


$\Rightarrow$



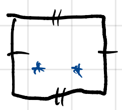
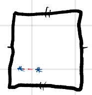
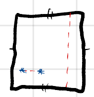
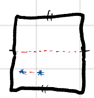
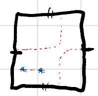
Some correlators

1) two point function on sphere:

$\langle \sigma \sigma \rangle_{S^2} = \Big\langle$  $\Big\rangle = \frac{1}{d_\sigma} \Big\langle$  $\Big\rangle = \frac{1}{d_\sigma} \Big\langle$  $\Big\rangle = \frac{1}{d_\sigma} \Big\langle$  $\Big\rangle = \frac{1}{d_\sigma} \Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle = \langle \mu \mu \rangle_{S^2}$

$\Rightarrow \langle \sigma \sigma \rangle_{S^2} = \langle \mu \mu \rangle_{S^2}$ Kramers-Wannier duality !!!

2) two point function on torus:

$\langle \sigma \sigma \rangle_{T^2}$:  $\sim$  $+$  $+$  $+$ 

Extra terms from fusion rules