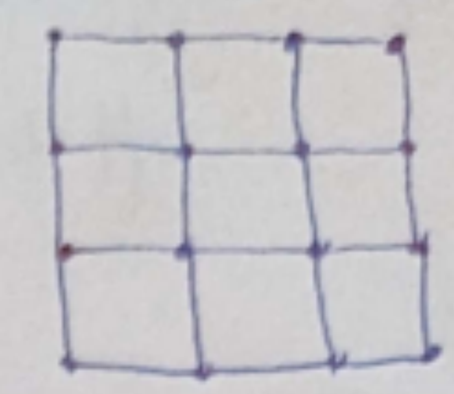
Ising model on square lattice.

- $S = \{0,1,\ldots,N\} \times \{0,1,\ldots,N\} \subseteq \Lambda = \mathbb{Z}\times\mathbb{Z} \subseteq \mathbb{R}^2$ "sites".



- $E = \{$pairs of adjacent sites$\}$ "links"/ "edges" $\langle ij\rangle$, $i,j \in S$.
- a spin configuration is $\sigma : S \longrightarrow \{\pm 1\}$. $\sigma_i = \sigma(i)$.
- energy : $H(\sigma) = -J\sum_{\langle ij\rangle} \sigma_i\sigma_j$ more generally, $H(\sigma) = -\sum_{\langle ij\rangle} J_{ij}\,\sigma_i\sigma_j$.
- probability distribution (Boltzmann) on $\{\sigma : S \to \{\pm1\}\}$, $P(\sigma) \propto e^{-\beta H(\sigma)}$
  - lowest energy = highest probability. "ground state" - all spins aligned.
  - β is the inverse temperature. $\left[\beta = \frac{1}{k_B T}\right]$.
- ~~to compute expectation values, e.g.~~ $\cancel{\langle H\rangle = \frac{\sum H(\sigma) e^{-H(\sigma)}}{\sum e^{-H(\sigma)}}}$, ~~good to know the denominator~~;
- partition fn : $Z(\beta) = \sum_{\{\sigma\}} e^{\beta J \sum_{\langle ij\rangle}\sigma_i\sigma_j}$. normalizing const., $P(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z(\beta)}$.

in the thermodynamic limit ($N \to \infty$), $Z(\beta)$ has a singularity - phase change.

- magnetization $\langle M\rangle = \frac{\sum M(\sigma) e^{-\beta H(\sigma)}}{Z(\beta)}$, $M(\sigma) = \frac{1}{N}\sum \sigma_i$, $\langle M\rangle \approx 0$ as $\beta \to 0$, $\langle M\rangle \approx 1$ as $\beta \to \infty$ for large N.

[Peierls]

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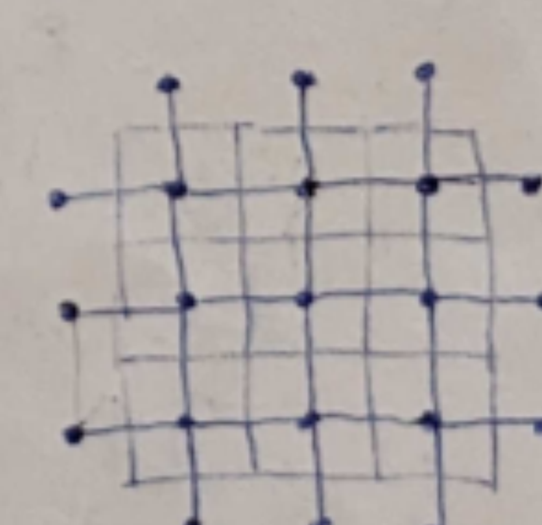
dual model.

- $\tilde{S} = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, N+\frac{1}{2}\} \times \{-\frac{1}{2}, \frac{1}{2}, \ldots, N+\frac{1}{2}\} \subseteq \tilde{\Lambda} = \Lambda + \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} \subset \mathbb{R}^2$. "dual sites".
- $\tilde{E} = \{$pairs of adjacent dual sites$\}$ excluding $\partial\tilde{S}$ "dual links / dual edges". $\tilde{E} \cong E$
- dual spin configuration $\tilde{\sigma} : \tilde{S} \to \{\pm1\}$, $\tilde{\sigma}\big|_{\partial\tilde{S}} = 1$.



- energy : $H(\tilde{\sigma}) = -J\sum_{\langle lm\rangle} \tilde{\sigma}_l\tilde{\sigma}_m$; $\tilde{Z}(\beta) = \sum_{\{\tilde{\sigma}\}} e^{\beta J\sum_{\langle lm\rangle}\tilde{\sigma}_l\tilde{\sigma}_m}$.

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duality:

$$Z(\beta) = f(\tilde{\beta})\,\tilde{Z}(\tilde{\beta})$$

, $f(\tilde{\beta})$ cancels out in expectation values.

$$\tilde{\beta} = -\tfrac{1}{2}\ln\tanh\beta$$

$$\langle \sigma_a\sigma_b\rangle = \frac{\tilde{Z}[\Gamma](\tilde{\beta})}{\tilde{Z}(\tilde{\beta})}$$

$$Z(\beta) = \sum_{\{\sigma\}} \exp\Big(\beta \sum_{\langle ij\rangle} \sigma_i\sigma_j\Big) \qquad [J=1]$$

$$= \sum_{\{\sigma\}} \prod_{\langle ij\rangle} e^{\beta\sigma_i\sigma_j}$$

$$= \sum_{\{\sigma\}} \prod_{\langle ij\rangle} C_0(\beta)\,(\sigma_i\sigma_j)^0 + C_1(\beta)\,(\sigma_i\sigma_j)^1$$

$$C_k(\beta) = (\cosh\beta \sinh\beta)^{\frac{1}{2}} \exp\Big(-\tfrac{1}{2}(1-2k)\ln\tanh\beta\Big) = \begin{cases} \cosh\beta & k=0 \\ \sinh\beta & k=1 \end{cases}.$$

Consider function $k: E \longrightarrow \{0,1\}$.

$$= \sum_{\{\sigma\}} \sum_{\{k\}} \prod_{\langle ij\rangle} C_{k_{ij}}(\beta)\,(\sigma_i\sigma_j)^{k_{ij}}$$

$$= \sum_{\{k\}} \Big(\prod_{\langle ij\rangle} C_{k_{ij}}(\beta)\Big)\Big(\sum_{\{\sigma\}} \prod_{\langle ij\rangle} (\sigma_i\sigma_j)^{k_{ij}}\Big)$$

$$= \sum_{\{k\}} \Big(\prod_{\langle ij\rangle} C_{k_{ij}}(\beta)\Big)\Big(\prod_{\text{sites } i} \sum_{\sigma_i=\pm1} \sigma_i^{\sum_j k_{ij}}\Big) \qquad \sum_{j:\langle ij\rangle} \text{ sum over neighbours } j \text{ of } i$$

$$= \sum_{\{k\}} \Big(\prod_{\langle ij\rangle} C_{k_{ij}}(\beta)\Big)\Big(\prod_i 2\,\delta_{\text{even}}\Big(\sum_{j:\langle ij\rangle} k_{ij}\Big)\Big). \qquad \delta_{\text{even}}(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

- any k such that $(*)\ \sum_{j:\langle ij\rangle} k_{ij}$ is even for all i is obtained from $\tilde\sigma$ by the following:

$$\tilde\sigma \longmapsto k^{\tilde\sigma} = \Big(k^{\tilde\sigma}_{ij} = \tfrac{1}{2}(1-\tilde\sigma_l\tilde\sigma_m)\Big),$$ where for $\langle ij\rangle$, $\langle lm\rangle$ is the dual edge dual to $\langle ij\rangle$.

[one may think of k as fn. on dual edges, $\tilde k_{lm} = k_{ij}$, w/ $\langle lm\rangle$ dual to $\langle ij\rangle$,
aka. a 1-cochain on $\tilde S$; the parity condition $(*)$ is equiv to $\delta\tilde k = 0$.
since there is no topology, $\tilde k$ must be exact; $(H^1(\tilde S, \partial\tilde S) = 0)$
and indeed, $k^{\tilde\sigma} = \delta\tilde\sigma$ essentially.]

$$Z(\beta) = \sum_{\{\tilde\sigma\}} \Big(\prod_{\langle ij\rangle} C_{k^{\tilde\sigma}_{ij}}(\beta)\Big) 2^N, \qquad N = |S|$$

$$= 2^N (\cosh\beta\sinh\beta)^{|E|/2} \sum_{\{\tilde\sigma\}} \prod_{\langle lm\rangle} \exp\Big(-\tfrac{1}{2}(\tilde\sigma_l\tilde\sigma_m)\ln\tanh\beta\Big)$$

$$= 2^N (2\sinh 2\tilde\beta)^{-|E|/2} \sum_{\{\tilde\sigma\}} \exp\Big(\tilde\beta \sum_{\langle lm\rangle} \tilde\sigma_l\tilde\sigma_m\Big) \qquad \text{where } \tilde\beta = -\tfrac{1}{2}\ln\tanh\beta.$$

$$\Big[\text{and } \cosh\beta\sinh\beta = \frac{1}{2\sinh 2\tilde\beta}\Big].$$

fix sites $a, b \in S$.

$$\langle \sigma_a \sigma_b \rangle = \sum_{\{\sigma\}} \sigma_a \sigma_b \exp\left(\beta \sum \sigma_i \sigma_j\right) \Big/ Z(\beta).$$

$$= \dots$$

$$= \sum_{\{k\}} \left(\prod_{\langle ij \rangle} c_{k_{ij}}(\beta) \right) \left(\sum_{\{\sigma\}} \sigma_a \sigma_b \prod_{\langle ij \rangle} (\sigma_i \sigma_j)^{k_{ij}} \right).$$

$$= \sum_{\{k\}} \left(\prod_{\langle ij \rangle} c_{k_{ij}}(\beta) \right) \left(\prod_{i \neq a,b} 2\delta_{\text{even}}\Big(\sum_{j:\langle ij \rangle} k_{ij} \Big) \right) 2\delta_{\text{even}}\Big(1 + \sum_{j:\langle aj \rangle} k_{aj}\Big)\, 2\delta_{\text{even}}\Big(1 + \sum_{j:\langle bj \rangle} k_{bj}\Big)$$

· want to obtain k satisfying parity conditions from $\tilde\sigma$, as before, but slightly modified:

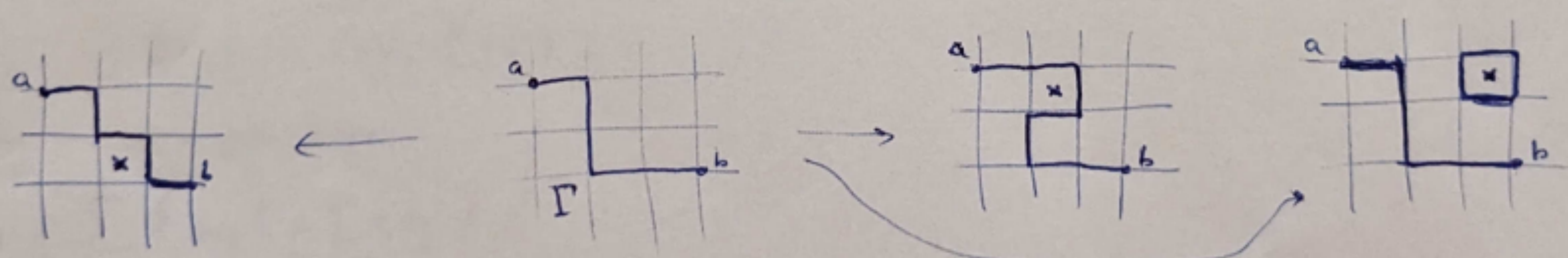
- choose path $\Gamma : i \to j$ on Λ; then define

$$\tilde\sigma \longmapsto \left(k^{\tilde\sigma}_{ij} = \begin{cases} \frac{1}{2}(1 - \cancel{2}\,\tilde\sigma_\ell \tilde\sigma_m) & ij \notin \Gamma \\ \frac{1}{2}(1 + \tilde\sigma_\ell \tilde\sigma_m) & ij \in \Gamma \end{cases} \right. \qquad \left(k^{\tilde\sigma}_{ij} \to 1 - k^{\tilde\sigma}_{ij} \text{ for } ij \text{ on } \Gamma \right)$$

$$= \underbrace{2^N (2\sinh 2\tilde\beta)^{-|E|/2}}_{\text{common w/ } \tilde Z(\tilde\beta)} \underbrace{\sum_{\tilde\sigma} \exp\Big(\tilde\beta \overbrace{\sum_{\langle \ell m \rangle} \tilde J_{\ell m} \tilde\sigma_\ell \tilde\sigma_m}^{H[\Gamma]\cancel{\beta}(\tilde\sigma)}\Big)}_{=: \tilde Z[\Gamma](\tilde\beta)}, \qquad \tilde J_{\ell m} = \begin{cases} +1 & \ell m \not\cap \Gamma \\ -1 & \ell m \cap \Gamma \end{cases}$$

— partition fn. for dual Ising model w/ defect along Γ.

$$\langle \sigma_a \sigma_b \rangle = \frac{\tilde Z[\Gamma](\tilde\beta)}{\tilde Z(\tilde\beta)}$$



invariance wrt. Γ

$$\Gamma \longrightarrow \Gamma' = \Gamma + \partial x$$

$$\tilde\sigma \longmapsto \tilde\sigma' : \ell \mapsto (-1)^{\delta_{\ell x}} \tilde\sigma_\ell.$$ ie. flip $\tilde\sigma_x$.

$$H[\Gamma](\tilde\sigma) = H[\Gamma'](\tilde\sigma').$$

and in particular, $\tilde Z[\Gamma](\tilde\beta) = \tilde Z[\Gamma'](\tilde\beta)$.

$$\langle \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_k} \rangle = \begin{cases} 0 & k \text{ odd} \quad \{\text{no } k \text{ can satisfy parity conditions} \\ \tilde Z[\Gamma_1, \dots, \Gamma_{k/2}](\tilde\beta) \big/ \tilde Z(\tilde\beta), & \Gamma_1, \dots, \Gamma_{k/2} \text{ are paths pairing up } a_i\text{'s.} \end{cases}$$

3D, 4D Ising.

$S = \{0, 1, \dots, N\}^3 \subset \Lambda \subset \mathbb{R}^3$.

$E = \{\text{pairs of adjacent sites}\}$.

$\sigma : S \longrightarrow \{\pm\}$, $H(\sigma) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$

dual to $\mathbb{Z}/2$ gauge theory on dual lattice.

$\tilde{S} = \{-\tfrac{1}{2}, \tfrac{1}{2}, \dots, N+\tfrac{1}{2}\}^3 \subset \tilde{\Lambda} = \Lambda + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \subset \mathbb{R}^3$.

$\tilde{E} = \{\text{pairs of adjacent dual sites}\}$.

$\tilde{P} = \{\text{dual plaquettes}\}$ i.e. ~~2~~ squares of $\tilde{\Lambda}$ in $\tilde{S}$. — 1-1 correspondence w/ E.

Consider $A : \tilde{E} \longrightarrow \{\pm 1\}$. $\left(A|_{\partial \tilde{S}} = +1 \right)$.

$$\tilde{H}(A) = -J \sum_{p \in \tilde{P}} \prod_{\langle ij \rangle \in \partial p} A_{ij}$$

; note $\tilde{H}(A) = \tilde{H}(A')$, where $A'_{ij} = \begin{cases} A_{ij} & \text{else} \\ -A_{ij} & \text{if } i=a \text{ or } j=a \end{cases}$

for some dual site a.

— i.e. gauge transformed.

$A' = A + \delta f$

$$\tilde{Z}(\beta) = \sum_{\{A\}}{}' \exp(-\beta \tilde{H}(A)).$$

Here $\sum'_{\{A\}}$ is sum over gauge-equiv. classes of A's.

Duality:

$$Z(\beta) = f(\tilde{\beta}) \tilde{Z}(\tilde{\beta})$$

essentially the same procedure.

$$Z(\beta) = \sum_{\{\sigma\}} \exp\left(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j\right)$$

$= \dots$

$$= \sum_{\{k\}} \left(\prod_{\langle ij \rangle} c_{k_{ij}}(\beta) \right) \left(\sum_{\{\sigma\}} \prod_{\langle ij \rangle} (\sigma_i \sigma_j)^{k_{ij}} \right)$$

$$= \sum_{\{k\}} \left(\prod_{\langle ij \rangle} c_{k_{ij}}(\beta) \right) \left(\prod_i 2 \delta_{\text{even}} \left(\sum_{j : \langle ij \rangle} k_{ij} \right) \right)$$

— any $k : E \to \{\pm 1\}$ satisfying the parity conditions arises from k^A

$$A \longmapsto k^A_{ij} = \tfrac{1}{2}\left(1 - \prod_{\partial p} A\right)$$ where p is the dual plaquette dual to ij.

— A is unique up to gauge transform, by similar cohomology arguments.

$$= \sum_{\{A\}}{}' \left(\prod_{\langle ij \rangle} c_{k^A_{ij}}(\beta) \right) 2^N = 2^N (2 \sinh 2\tilde{\beta})^{-|E|/2} \sum_{\{A\}}{}' \exp\left(\tilde{\beta} \sum_{p \in \tilde{P}} \prod_{\partial p} A \right)$$

$$\langle \sigma_a \sigma_b \rangle = \frac{\tilde{Z}[\Gamma](\tilde\beta)}{\tilde{Z}(\tilde\beta)}$$, proven similarly.

$$\langle \sigma_a \sigma_b \rangle = \sum_{\{\cancel{A}\sigma\}} \sigma_a \sigma_b \exp\Big(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j\Big) \Big/ Z(\beta)$$

$$= \sum_{\{k\}} \Big(\prod_{\langle ij \rangle} c_{k_{ij}}(\beta) \Big) \Big(\prod_{i \neq a,b}^{*} 2\delta_{\text{even}} \Big(\sum_{j:\langle ij \rangle} k_{ij} \Big) \Big) 2\delta_{\text{even}} \Big(1 + \sum_{j:\langle aj \rangle} k_{aj} \Big) 2\delta_{\text{even}} \Big(1 + \sum_{j:\langle bj \rangle} k_{bj} \Big).$$

$$A \mapsto k_{ij}^{A} = \begin{cases} \frac{1}{2}\big(1 - \prod_{\partial p} A\big) \\ \frac{1}{2}\big(1 + \prod_{\partial p} A\big) \end{cases} \quad p \text{ dual to } ij.$$

$$= \underbrace{2^{N} (2 \sinh 2\tilde\beta)^{-|E|/2}}_{\substack{\text{common } \cancel{w}/ \\ \tilde{Z}(\tilde\beta)}} \underbrace{\sum_{\{A\}}{}' \exp\Big(\tilde\beta \overbrace{\sum_{p \in \tilde{P}} J_p \prod_{\partial p} A}^{\tilde{H}[\Gamma](A)} \Big)}_{\tilde{Z}[\Gamma](\cancel{A}\,\tilde\beta)}, \quad J_p = \begin{cases} +1 & p \not\cap \Gamma \\ -1 & p \cap \Gamma \end{cases}.$$

<u>4D</u>

$S = \{0, \dots, N\}^4$, E,

dual:

$\tilde{S} = \{-\frac{1}{2}, \dots, N + \frac{1}{2}\}^4$, $\tilde{E}$, $\tilde{P}$, $\tilde{C}$ = dual cubes $\leftrightarrow E$.

$B : \tilde{P} \longrightarrow \{\pm 1\}$, $\quad \tilde{H}(B) = -J \sum_{c \in \tilde{C}} \prod_{p \in \partial c} B_p$. "higher gauge field"

— invariant under $B \mapsto B'_p = \cancel{(-1)} \begin{cases} B_p & e \notin \partial p \\ -B_p & e \in \partial p \end{cases}$ for some e,

$\tilde{Z}(\beta) = \sum_{\{B\}}{}' \exp\big(\cancel{\Sigma\Sigma} - \beta \tilde{H}(B)\big)$. $\quad$ or more generally $B' = B + \delta A$.

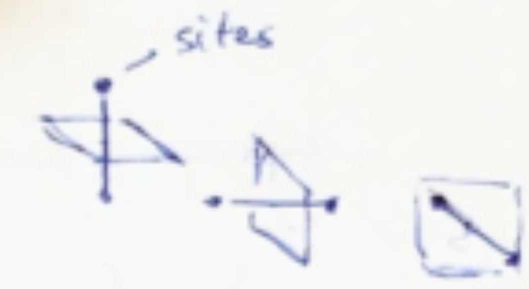
similar duality statements hold.

duality in 3D, gauge thing to spins in Ising.

$\tilde{P} = \{\text{dual plaquettes}\}$

$$\tilde{Z}(\beta) = \sum_A \exp(-\beta \tilde{H}(A))$$

$$= \sum_A \prod_{p \in \tilde{P}} \underbrace{\exp(-\beta \prod_{e \in \partial \tilde{p}} A_e)}$$



$\tilde{p}$ dual to $e \in E$.

$$= \dots \qquad C_0(\beta)(\prod A)^0 + C_1(\beta)(\prod A)^1$$

$$= \sum_{\substack{\tilde{k}: \tilde{P} \to \{0,1\} \\ \tilde{k}|_\partial = 0}} \Big(\prod_{\tilde{p}} C_{\tilde{k}_{\tilde{p}}} \Big) \Big(\prod_{\tilde{e} \in \tilde{E} \setminus \partial} 2\, \delta_{\text{even}} \Big(\sum_{\tilde{p}: \tilde{e} \in \partial \tilde{p}} \tilde{k}_{\tilde{p}} \Big) \Big)$$

parity condition means $\partial \tilde{k} = 0$

define $k: E \to \{0,1\}$, $k_e = \tilde{k}_{\tilde{p}}$ for $e \cap \tilde{p}$ dual.

$\partial \tilde{k} = 0 \iff \delta k = 0 \iff \exists \sigma,\ k = \delta \sigma$.

σ unique up to const. ie global spin flip.

$= \cancel{\dots}$

$$= \frac{1}{2} \sum_\sigma \Big(\prod_{\langle ij \rangle} (2 \sinh 2\tilde{\beta})^{\cancel{-1/2}} \exp(\tilde{\beta}\, \sigma_i \sigma_j) \Big)\, 2^{|\tilde{E} \setminus \partial|}$$

$$= \frac{1}{2}\, 2^{|\tilde{E} \setminus \partial|} (2 \sinh 2\beta)^{-|E|/2} \underbrace{\sum_\sigma \exp(\tilde{\beta} \sum \sigma_i \sigma_j)}_{Z(\tilde{\beta})}$$

$\tilde{\gamma}$ set of dual edges $\subseteq \tilde{E}$. Want to compute $\langle \tilde{\gamma} \rangle$ — only makes sense if $\tilde{\gamma}$ is invariant under gauge transform. [more precisely, $\prod_{\tilde{e} \in \tilde{\gamma}} A_{\tilde{e}}$ is invariant]

$$\langle \tilde{\gamma} \rangle_\beta = \frac{1}{\tilde{Z}(\beta)} \sum_A \exp(-\beta \tilde{H}(A)) \cdot \prod_{\tilde{e} \in \tilde{\gamma}} A_{\tilde{e}}$$

$\iff \tilde{\gamma}$ is a closed, $\partial \tilde{\gamma} = 0$.

e.g. a closed loop.

$$= \dots$$

$$= \frac{1}{\tilde{Z}} \sum_{\tilde{k}} \Big(\prod_{\tilde{p}} C_{\tilde{k}_{\tilde{p}}} \Big) \Big(\prod_{\tilde{e} \notin \tilde{\gamma}} 2\, \delta_{\text{even}} \Big(\sum_{\tilde{p}: \tilde{e} \in \partial \tilde{p}} \tilde{k}_{\tilde{p}} \Big) \Big) \cdot \Big(\prod_{\tilde{e} \in \tilde{\gamma}} 2\, \delta_{\text{even}} \Big(1 + \sum_{\tilde{e} \in \partial \tilde{p}} \tilde{k}_{\tilde{p}} \Big) \Big)$$

- choose surface [made of dual plaquettes] $\tilde{X}$, such that $\partial \tilde{X} = \tilde{\gamma}$.

$$(\tilde{k}^{\tilde{X}})_{\tilde{p}} = \begin{cases} 0 & \tilde{p} \notin \tilde{X} \\ 1 & \tilde{p} \in \tilde{X} \end{cases} = \chi_{\tilde{X}} \text{ indicator fn.}$$

— has the required parity condition

use assignment $\sigma \mapsto \delta \sigma + \tilde{k}^{\tilde{X}}$.

$$(-1)^{\langle ij \rangle \cap \tilde{X}} = \begin{cases} +1 & \cap \\ -1 & \not\cap \end{cases}$$

$$= \dots$$

$$= \frac{1}{2} \frac{1}{\tilde{Z}(\beta)} \sum_\sigma \Big(\prod_{\langle ij \rangle \not\cap \tilde{X}} \exp(\tilde{\beta} \sigma_i \sigma_j) \Big) \Big(\prod_{\langle ij \rangle \cap \tilde{X}} \exp(-\tilde{\beta} \sigma_i \sigma_j) \cdot (2 \sinh 2\tilde{\beta})^{-|E|/2} \Big) \cdot 2^{|\tilde{E} \setminus \partial|}$$

$$= 4\, \frac{Z[\tilde{X}](\tilde{\beta})}{Z(\tilde{\beta})}$$

, $Z[\tilde{X}]$ being partition fn for Ising model w/ defect along $\tilde{X}$, $H[\tilde{X}](\sigma) = -\sum_{\langle ij \rangle} (-1)^{\langle ij \rangle \cap \tilde{X}} \sigma_i \sigma_j$