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TFT CONSTRUCTION OF RCFT CORRELATORS I: PARTITION FUNCTIONS

Jürgen Fuchs [1] Ingo Runkel [2] Christoph Schweigert [2]

[1] Institutionen för fysik
Universitetsgatan 5
S – 651 88 Karlstad

[2] LPTHE, Université Paris VI
4 place Jussieu
F – 75 252 Paris Cedex 05

Abstract

We formulate rational conformal field theory in terms of a symmetric special Frobenius algebra A and its representations. A is an algebra in the modular tensor category of Moore–Seiberg data of the underlying chiral CFT. The multiplication on A corresponds to the OPE of boundary fields for a single boundary condition. General boundary conditions are A-modules, and (generalised) defect lines are A-A-bimodules.

The relation with three-dimensional TFT is used to express CFT data, like structure constants or torus and annulus coefficients, as invariants of links in three-manifolds. We compute explicitly the ordinary and twisted partition functions on the torus and the annulus partition functions. We prove that they satisfy consistency conditions, like modular invariance and NIM-rep properties.

We suggest that our results can be interpreted in terms of non-commutative geometry over the modular tensor category of Moore–Seiberg data.

1 Introduction and summary

The use of two-dimensional conformal field theory in string theory, statistical mechanics and condensed matter physics has often focussed on issues related to bulk fields on closed world sheets. But among the multitude of applications of CFT there are also many – like the study of percolation probabilities, of defects in condensed matter systems and of string perturbation theory in D-brane backgrounds – that require an understanding of CFT on world sheets with boundary, and in particular of conformally invariant boundary conditions. These aspects have been investigated intensively over the last few years. Apart from its important physical applications, the study of boundary conditions is also considerably contributing to increase our structural insight in conformal field theory. Further progress can be expected to result from the analysis of defect lines, a subject that so far has attracted comparatively moderate attention.

In the present paper we make transparent the behavior of rational conformal field theories on arbitrary (orientable) world sheets, including boundaries and defect lines. This is achieved by combining tools from topological field theory with concepts from non-commutative algebra, making ample use of two basic facts:

- The Moore–Seiberg data of a rational chiral CFT give rise to a topological field theory in three dimensions, and thereby to invariants of links in three-manifolds.
- The Moore–Seiberg data give rise to a modular tensor category $\mathcal{C}$. One can set up algebra and representation theory in this category $\mathcal{C}$ in very much in the same way as it is usually done in the categories of vector spaces or of super-vector spaces.

A modular tensor category is actually nothing else than a basis-independent formulation of the Moore–Seiberg data. An important motivation to adopt this framework is the observation that there exist several rather different mathematical formalisations of the physical intuition of a chiral conformal field theory, i.e. of the chiral algebra, its space of physical states and of the properties of chiral vertex operators and conformal blocks associated to these states. Two prominent examples of such formalisms are the one based on local algebras of observables on the circle, and hence nets of subfactors, and the one of vertex algebras. Both frameworks involve quite intricate mathematical structures. Accordingly, in both settings the explicit treatment of even modestly complicated models proves to be difficult.

A major problem is to work out the representation theory of the vertex algebras, respectively to find the (physically relevant) representations of the local algebras of observables. As a consequence, there have been various attempts to extract the relevant part of the information about the representation category of the chiral algebra and to encode it in simpler structures. These attempts have been particularly successful for rational theories, for which the representation category is semisimple and there are only finitely many inequivalent irreducible representations. In the present paper we require the chiral algebra to be rational. However, we do not insist on choosing the maximally extended chiral algebra. This allows us to deal also with symmetry breaking boundary conditions, as well as theories for which the left- and right-moving chiral algebras are different.

The attempts to formalise aspects of the representation theory of rational conformal field theories have lead, among other results, to new algebraic notions, like truncated quantum groups (see e.g. [1, 2, 3, 4, 5]), weak Hopf algebras [6, 7, 8, 9] and double triangle algebras [10, 11]. A more direct approach is to formalise the properties of the representation category itself. This gives rise to the notion of a modular tensor category [12], which we will explain in detail in section 2, and of module categories [13]. Schematically:

formalisations of the chiral algebra	structures capturing the representation category
nets of subfactors	truncated quantum groups
	weak Hopf algebras
vertex algebras	double triangle algebras
	modular tensor categories

By making use of double triangle algebras and weak Hopf algebras, aspects of rational CFT have been analyzed in [14, 15, 16, 17]. The present paper develops an approach to rational CFT that is based on modular categories.

The Moore–Seiberg data captures the *chiral* aspects of rational conformal field theory. To arrive at a *full* conformal field theory with local correlation functions, additional input is required. This can already be seen in the example of a free boson. Here one can choose $\hat{\mathfrak{u}}(1)$ as the chiral algebra, which has a unitary irreducible highest weight representation for every charge $q \in \mathbb{R}$. There are many consistent CFTs associated to these chiral data, for example those describing a free boson compactified on a circle of some given radius. These models possess (modulo T-duality) in particular different modular invariant torus partition functions. So the first structure that comes to mind as additional information for the construction of a full conformal field theory is the choice of a modular invariant for the bulk theory. However, this proves to be too naive, as one knows of many examples of modular invariant bilinear combinations of characters that do not arise as the partition function of any consistent CFT at all [18, 19, 20, 21].

It is therefore a crucial insight [22] that complete information on how to construct a full CFT from given chiral data is contained in the structure of a *symmetric special Frobenius algebra* in $\mathcal{C}$. (We will explain in detail below what is meant by an algebra A in $\mathcal{C}$. It is an object, with specific properties, in the category $\mathcal{C}$, and thereby corresponds to some representation of the chiral algebra – the algebra object A must in particular not be confused with the chiral algebra $\mathfrak{V}$ itself.) Already at this point a significant advantage of tensor categories becomes apparent: Once one accepts the idea of doing algebra and representation theory in the setting of general tensor categories rather than the category of complex vector spaces, one can directly use standard algebraic and representation theoretic concepts, like, in the case at hand, the notion of a Frobenius algebra. Indeed, all mathematical concepts that are needed in the approach to CFT that is developed here can already be found in standard textbooks on associative algebras [23, 24] and category theory [25, 26].

We can show that every symmetric special Frobenius algebra object in the modular tensor category of a chiral CFT leads to a full CFT that is consistent on all orientable world sheets; Morita equivalent algebras yield the same CFT. Conversely, we establish that every unitary rational full CFT – provided only that it possesses one boundary condition preserving the chiral algebra [1] at all – determines uniquely a (Morita class of) symmetric special Frobenius algebra(s). The Frobenius algebra in question is actually nothing else than the algebra of

[1] Since the chiral algebra is not required to be maximally extended, the boundary condition is still allowed to break part of the bulk symmetry. For brevity, in this paper we will sometimes refer to boundary conditions that preserve the chiral algebra as 'conformal boundary conditions' or also just as 'boundary conditions'.

boundary fields associated to one given boundary condition of the theory. It is associative due to the associativity of the operator product of boundary fields; the non-degenerate bilinear invariant form that turns it into a Frobenius algebra expresses the non-degeneracy of the two-point functions of boundary fields on the disk. Our results can thus be briefly summarised by saying that we are able to *construct the correlation functions of a unitary rational conformal field theory starting from just one of its boundary conditions*.

In fact, every boundary condition of a full CFT gives rise to a symmetric special Frobenius algebra object, and all such algebra objects are Morita equivalent and hence lead to one and the same CFT. Moreover, when a given full CFT can be constructed from a Frobenius algebra A, then any of its boundary conditions gives back an algebra in the Morita class of A. (On the other hand, we cannot, as yet, exclude the possibility that there exists a boundary condition M of some CFT C such that the CFT reconstructed from the Frobenius algebra A that arises from M does not coincide with the original CFT C.)

Apart from its conceptual aspects, the formalism presented in this paper is also of considerable practical and computational value. The main point is that *structure constants* – OPE constants as well as coefficients of the torus and annulus partition functions – *are given as link invariants in three-manifolds*. Computing the value of an invariant is straightforward once we have gathered three ingredients: the Moore–Seiberg data (i.e. the fusing and braiding matrices), the structure constants for the multiplication of the algebra object A, and the representation matrices describing the action of A on its irreducible modules.

We will treat the Moore–Seiberg data as given. It is of course a non-trivial problem to obtain these data from a chiral CFT; here, however, we are concerned with building a full CFT given all the chiral information. Finding an algebra and a multiplication involves solving a nonlinear associativity constraint. It turns out that this constraint is equivalent to the sewing constraint for boundary structure constants of a *single* boundary condition. Solving this nonlinear equation is not easy, but still much simpler than finding a solution to the full set of nonlinear constraints involving all boundary structure constants as well as the bulk-boundary couplings and the bulk structure constants. Finally, finding the representations of A is a *linear* problem.

There is a concept that allows us to systematically construct examples of symmetric special Frobenius algebras: simple currents [27], that is, the simple objects of $\mathcal{C}$ with quantum dimension one. As we will explain elsewhere, for algebra objects that contain only simple currents as simple subobjects the associativity constraints reduce to a cohomology problem for abelian groups that can be solved explicitly. Algebras built from simple currents describe modular invariants of 'D-type'. Often, in particular for all WZW models, they provide representatives for almost all Morita classes of algebras. It is, however, a virtue of the formalism developed in this paper that it treats exceptional modular invariants, including those of automorphism type, on the same footing as simple current modular invariants. (The structure of full conformal field theories having an exceptional modular invariant is therefore not really exceptional.)

An important aspect of our construction is that it is possible to prove that the resulting structure constants of the CFT solve all sewing constraints. From a computational point of view, the present formalism thus allows us to generate a solution to the full set of sewing constraints from a solution to a small subset of these constraints. This way one can also check the consistency of boundary conditions that have been proposed in the literature.

Modular tensor categories possess in particular a braiding which accounts for the braid group statistics of two-dimensional field theories. Thus there is a natural notion of commutativity with respect to this braiding. It is therefore worth emphasising that the Frobenius algebras A we consider are not necessarily (braided-) commutative. As a consequence, our construction constitutes a natural generalisation of *non-commutative* algebra to modular tensor categories. This allows us to summarise our results in the following dictionary between physical concepts in CFT and notions in the theory of associative algebras:

physical concepts	algebraic notions
boundary conditions	A-modules
defect lines	A-A-bimodules
boundary fields Ψ_i^{MN}	$\mathrm{Hom}_A(M\otimes i, N)$
bulk fields Φ_{ij}	$\mathrm{Hom}_{A\vert A}((A\otimes j)^-, (A\otimes \bar\imath)^+)$
disorder fields $\Phi_{ij}^{B_1,B_2}$	$\mathrm{Hom}_{A\vert A}((B_1\otimes j)^-, (B_2\otimes \bar\imath)^+)$

Let us explain the entries of this table in some detail. Boundary conditions will be shown to be in correspondence with (left) modules of A. In particular, simple modules correspond to elementary boundary conditions, while direct sums of simple modules indicate the presence of Chan–Paton multiplicities. If M is a left A-module, then for any object i of $\mathcal{C}$, $M\otimes i$ is a left A-module, too. As a consequence, it makes sense to consider left A-module morphisms from $M\otimes i$ to another module N, i.e. morphisms from $M\otimes i$ to N that intertwine the action of A on the two objects. For each such morphism there is a boundary field changing the boundary condition from M to N and carrying the chiral label i.

For non-commutative algebras, it is natural to consider not only left (or right) modules, but also bimodules, i.e. objects that carry an action of A both from the left and from the right, such that both actions commute. We show that bimodules correspond to (generalised) defect lines. The trivial defect line – i.e. no defect at all – is A itself, and the tensor product (over A) of bimodules is 'fusion' of defect lines.

Given a bimodule B_1, one can endow the object $B_1\otimes i$ for any object i of $\mathcal{C}$ with the structure of a bimodule in two different ways: The left action of A is just the one inherited from the left action of A on B_1, while a right action of A can be defined by using either the braiding of i and A or the inverse of this braiding. We denote the two resulting bimodules by $(B_1\otimes i)^+$ and $(B_1\otimes i)^-$.

A particular bimodule is A itself. The degeneracy of a bulk field with chiral labels i and j is again given by a space of morphisms – the space $\mathrm{Hom}_{A|A}((A\otimes j)^-, (A\otimes \bar\imath)^+)$ of bimodule morphisms. This suggests the following re-interpretation of bulk fields: they "change" the trivial defect A to itself. It is therefore natural to generalise bulk fields and consider fields with chiral labels i, j that change the defect line of type corresponding to the bimodule B_1 to a defect line of some other type B_2. The degeneracy of these disorder fields is described by the space $\mathrm{Hom}_{A|A}((B_1\otimes j)^-, (B_2\otimes \bar\imath)^+)$ of bimodule morphisms.

For each such type of fields there are partition functions that count the corresponding states. For boundary fields these are linear, for bulk fields bilinear combinations of characters with non-negative integral coefficients. This way, every full rational conformal field theory gives rise to a collection of combinatorial data – essentially the dimensions of the morphism spaces introduced in the table above. Clearly, these data must satisfy various consistency constraints, both among each other and with the underlying category $\mathcal{C}$, in particular with the fusion rules of $\mathcal{C}$. Concrete instances of such consistency conditions have been obtained from a variety of arguments, see e.g. [28, 29, 30, 31, 32, 14, 33, 34, 35, 36, 21]. In particular, the annulus partition functions provide non-negative integral matrix representations (NIM-reps) of the fusion rules of $\mathcal{C}$, while the partition functions of defect line changing operators give rise to NIM-reps of the double fusion rules.

One important result of the present paper is a rigorous proof of these relations. As a word of warning, let us point out that, by themselves, the problems of classifying modular invariants or NIM-reps are *not physical* problems. (Still, the classification of such combinatorial data can be a useful auxiliary task.) Indeed, as already mentioned, they tend to possess solutions that do not describe the partition functions of any conformal field theory (see e.g. [18,19,20,21]). In contrast, in our approach, the partition functions arise as special cases of correlation functions. Indeed, in a forthcoming publication our approach will be extended to general amplitudes (compare [22]), and it will be shown that the system of amplitudes, with arbitrary insertions and on arbitrary world sheets, satisfies all factorisation and locality constraints, and that they are invariant under the relevant mapping class groups. This result guarantees that only physical solutions occur in our approach.

A brief outline of the paper is as follows. We start in section 2 with a review of some facts about modular tensor categories and topological field theory. In sections 3 and 4 we investigate symmetric special Frobenius algebra objects and their representation theory, respectively, and show how these structures arise in conformal field theory. In the remainder of the paper these tools are employed to deduce various properties of torus and annulus partition functions and to study defect lines. As an illustration how our approach works in practice, two examples accompany our development of the general theory: The free boson compactified at a radius of rational square, and the E_7 modular invariant of the $\mathfrak{su}(2)$ WZW model.

Some of our results have been announced in [22, 37].

Contents

2 Modular tensor categories and chiral CFT

2.1 Modular tensor categories

As already pointed out in the introduction, the framework we are going to use is the one of modular tensor categories [12, 26, 38, 39]. Let us explain in some detail what these structures are and why it is natural and appropriate to work in this setting.

In conformal field theory, modular tensor categories arise in the form of representation categories of rational vertex operator algebras [40, 41, 42], [2] which in turn constitute a concrete mathematical realisation of the physical concepts of a chiral algebra and its primary fields. It has been demonstrated by Moore and Seiberg [46, 47] that the basic properties of a rational chiral conformal field theory can be encoded in a small collection of data – like braiding and fusing matrices and the modular S-matrix – and relations among them – like the pentagon and hexagon identities. One must be aware, however, of the fact that the usual presentation of those data implicitly involves various non-canonical basis choices. As a consequence, the fusing matrices, for instance, enjoy a large gauge freedom, whereas only their gauge-invariant part has a physical meaning. Posing the Moore–Seiberg data in a basis-free setting leads rather directly to the concept of a modular tensor category. As an additional benefit, this formulation supplies us with a powerful graphical calculus.

In the sequel we start out by reviewing details of the mathematical machinery that is required to understand modular tensor categories; only afterwards we return to the origin of these structures in rational conformal field theory. A *category* $\mathcal{C}$ consists of two types of data: A class $\mathcal{O}bj(\mathcal{C})$ of *objects* and a family of *morphism* sets $\mathrm{Hom}(U,V)$, one for each pair $U,V \in \mathcal{O}bj(\mathcal{C})$. In the situation of our interest, the objects are the representations of the chiral algebra $\mathfrak{V}$ of the CFT (a rational vertex operator algebra), and the morphisms are the intertwiners between $\mathfrak{V}$-representations.

Morphisms can be composed when the relevant objects match, i.e. the composition $g\circ f$ of $f\in\mathrm{Hom}(U,V)$ and $g\in\mathrm{Hom}(Y,Z)$ exists if $Y=V$. This operation of composition is associative, and for every object U the endomorphism space $\mathrm{End}(U)\equiv\mathrm{Hom}(U,U)$ contains a distinguished element, the identity morphism id_U, satisfying $g\circ id_U=g$ for all $g\in\mathrm{Hom}(U,V)$ and $id_U\circ f=f$ for all $f\in\mathrm{Hom}(Y,U)$. The categories $\mathcal{C}$ of our interest are complete with respect to direct sums (this can always be assumed without loss of generality) and come enriched with quite a bit of additional structure; we introduce this structure in three steps.

■ First, $\mathcal{C}$ is a semisimple abelian strict tensor category with the complex numbers as ground ring.

Let us explain the various qualifications appearing in this statement. *Abelianness* [25, chapter VIII] means that there is a zero object 0 and the morphisms possess various natural properties familiar from vector spaces. Concretely, every morphism set is an abelian group, and composition of morphisms is bilinear; every morphism has a kernel and a cokernel (they are defined by a universal property); every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel; finally, every morphism f can be written as the composition $f=h\circ g$ of a monomorphism h and an epimorphism g.

[2] It is still a conjecture that the representation category of every rational VOA is modular. There is no general proof, but the property has been established for several important classes of VOAs, compare e.g. [43, 44, 45], and it is expected that possible exceptions should better be accounted for by an appropriate refinement of the qualification 'rational'.

In a *tensor* category (see e.g. [25, chapter VII] or [26, chapter XI]) there is a tensor product $\otimes$, both of objects and of morphisms. The tensor product on objects has a unit element, which is denoted by $\mathbf{1}$; for $f \in \mathrm{Hom}(U,Y)$ and $g \in \mathrm{Hom}(V,Z)$, the tensor product morphism $f \otimes g$ is an element of $\mathrm{Hom}(U{\otimes}V, Y{\otimes}Z)$. The endomorphisms $\mathrm{End}(\mathbf{1})$ of the tensor unit $\mathbf{1}$ form a commutative ring k, called the *ground ring*, and every morphism set is a k-module. The operations of composition and of forming the tensor product of morphisms are bilinear and compatible in an obvious manner. In the present context we require that the ground ring is the field of complex numbers, $k = \mathbb{C}$; then the morphism sets are complex vector spaces.
In any tensor category there is a family of isomorphisms between $U \otimes (Y{\otimes}Z)$ and $(U{\otimes}Y) \otimes Z$, with U, Y, Z any triple of objects, and families of isomorphisms between $U \otimes \mathbf{1}$ and U as well as between $\mathbf{1} \otimes U$ and U, for any object U. They are called associativity and unit constraints, respectively, and are subject to the so-called pentagon identity (assuring that any two possibilities of bracketing multiple tensor products are related via the associativity constraints) and triangle identities (compatibility between associativity and unit constraints). A tensor category is called *strict* if all these isomorphisms are identities, so that the tensor product of objects is associative and $\mathbf{1} \otimes U = U = U \otimes \mathbf{1}$. By the coherence theorems [25, chapter VII.2], there is no loss of generality in imposing this strictness property. On the other hand, when dealing with certain other structures below, we will often have to be careful not to mix up equality and isomorphy of objects.
Finally, the meaning of semisimplicity is analogous as for other algebraic structures. A *simple* (or irreducible) object U of an abelian tensor category is an object whose endomorphisms are given by the ground ring, $\mathrm{End}(U) = k\, id_U$, i.e. $\mathrm{End}(U) = \mathbb{C}\, id_U$ for the categories considered here;[3] in particular, the tensor unit $\mathbf{1}$ is automatically simple. A *semisimple* category is then characterised by the property that every object is the direct sum of finitely many simple objects.

Semisimplicity of a tensor category $\mathcal{C}$ implies in particular *dominance* of $\mathcal{C}$, which means that there exists a family $\{U_i\}_{i\in\mathcal{I}}$ of simple objects with the following property: for any $V, W \in \mathcal{O}bj(\mathcal{C})$ every morphism $f \in \mathrm{Hom}(V,W)$ can be decomposed into a finite sum

$$f = \sum_r g_r \circ h_r \tag{2.1}$$

with

$$h_r \in \mathrm{Hom}(V, U_i) \qquad \text{and} \qquad g_r \in \mathrm{Hom}(U_i, W) \tag{2.2}$$

for suitable members $U_i = U_i(r)$ (possibly with repetitions) of this family.

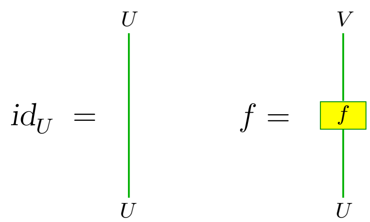
Since in the categories we are considering, the morphism sets are vector spaces (over $\mathbb{C}$), it is convenient to introduce a shorthand notation for their dimension:

$$\dim \mathrm{Hom}(X,Y) =: \langle X,Y \rangle \tag{2.3}$$

for $X, Y \in \mathcal{O}bj(\mathcal{C})$. As a consequence of semisimplicity we have $\langle X,Y \rangle = \langle Y,X \rangle$.

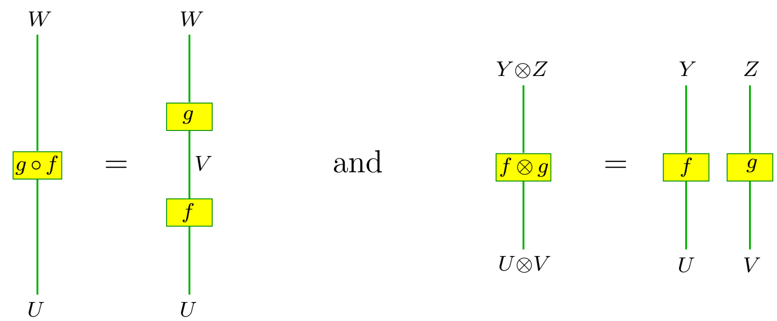
[3] In a general category, this property rather characterises an *absolutely simple* object, while simplicity of an object means that it does not possess a non-trivial proper subobject. In semisimple categories, absolutely simple implies simple, and in any abelian category over an algebraically closed ground field the two notions are equivalent.

A convenient way to visualise morphisms in an abelian tensor category is via graphs in which lines stand for identity morphisms; thus id_U and $f \in \mathrm{Hom}(U,V)$ are depicted as



(2.4)

We follow the convention that such pictures are read from bottom to top. Because of $\mathrm{End}(\mathbf{1}) = \mathbb{C}$ we have $id_{\mathbf{1}} = 1 \in \mathbb{C}$; accordingly, lines labelled by the tensor unit $\mathbf{1}$ can and will be omitted, so that in the pictorial description morphisms in $\mathrm{Hom}(\mathbf{1},U)$ or $\mathrm{Hom}(U,\mathbf{1})$ emerge from and disappear into 'nothing', respectively. Composition of morphisms amounts to concatenation of lines, while the tensor product corresponds to juxtaposition:



(2.5)

■ Second, $\mathcal{C}$ is a *ribbon category*, that is [26, chapter XIV.3], a strict tensor category supplemented with three additional ingredients: a duality, a braiding, and a twist. A (right) *duality* on a tensor [4] category $\mathcal{C}$ associates to every $U \in \mathcal{O}bj(\mathcal{C})$ another object $U^\vee \in \mathcal{O}bj(\mathcal{C})$ and morphisms

$$b_U \in \mathrm{Hom}(\mathbf{1}, U \otimes U^\vee)\,, \qquad d_U \in \mathrm{Hom}(U^\vee \otimes U, \mathbf{1})\,, \tag{2.6}$$

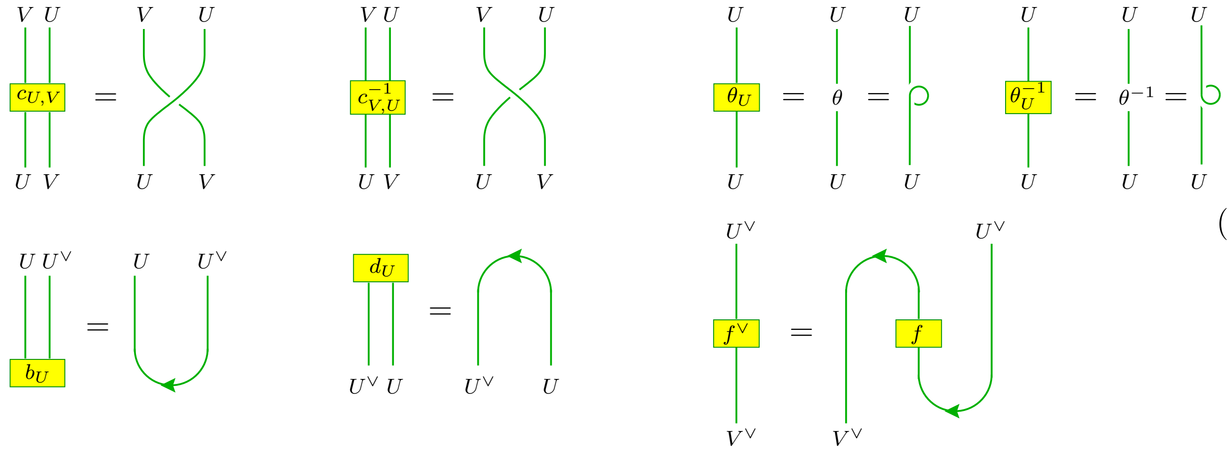
and to every morphism $f \in \mathrm{Hom}(U,Y)$ the morphism

$$f^\vee := (d_Y \otimes id_{U^\vee}) \circ (id_{Y^\vee} \otimes f \otimes id_{U^\vee}) \circ (id_{Y^\vee} \otimes b_U) \ \in \mathrm{Hom}(Y^\vee, U^\vee)\,. \tag{2.7}$$

$U^\vee$ is called the object (right-) dual to U, and $f^\vee$ the morphism (right-) dual to f; the duality morphisms d_U and b_U are also known as the evaluation and co-evaluation morphisms, respectively. A *braiding* on a tensor category $\mathcal{C}$ allows one to 'exchange' objects; it consists of a family of isomorphisms $c_{U,V} \in \mathrm{Hom}(U \otimes V, V \otimes U)$, one for each pair $U, V \in \mathcal{O}bj(\mathcal{C})$. Finally, a *twist* is a family of isomorphisms θ_U, one for each $U \in \mathcal{O}bj(\mathcal{C})$. Graphically, the braiding and twist and

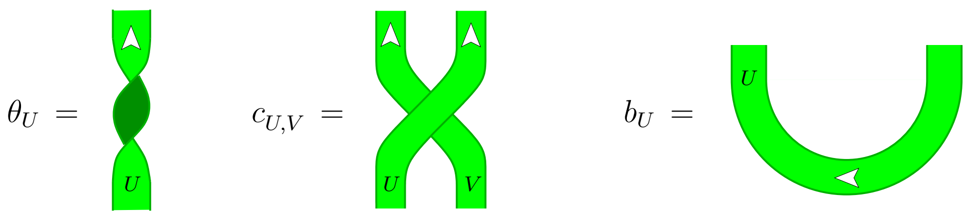
[4] Often the existence of a duality is included in the definition of the term 'tensor category'. What we refer to as a tensor category here is then called a *monoidal* category.

their inverses and the duality will be depicted as follows:



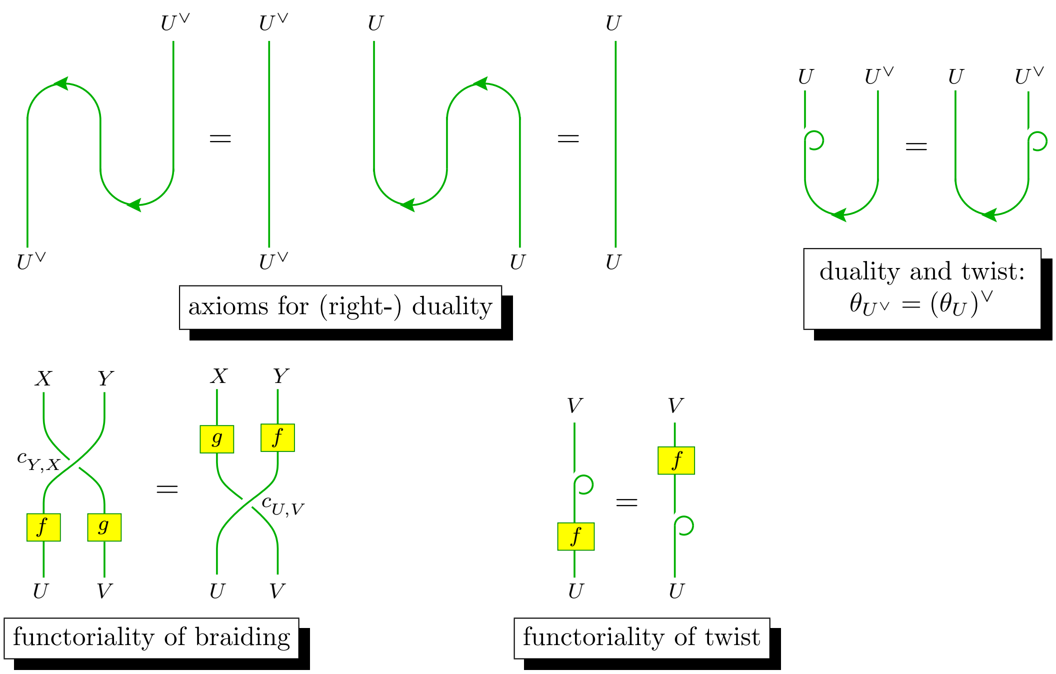
(2.8)

Actually we should think of these morphisms as ribbons rather than lines – this is the reason for the terminology. For example, the twist θ, braiding c and duality morphism b are drawn as

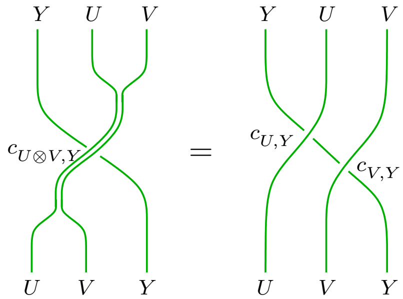


(2.9)

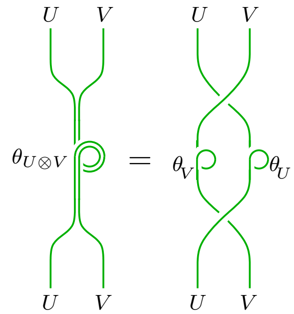
In the sequel the interpretation of graphs with lines as ribbon graphs will be implicit. The duality, braiding and twist are subject to a number of consistency conditions, which precisely guarantee that the visualisation via ribbons is appropriate, so that in particular the graphs obtained by their composition share the properties of the correspondingly glued ribbons. More concretely, one has to impose duality identities, functoriality and tensoriality of the braiding, functoriality of the twist, and compatibility of the twist with duality and with braiding. In the notation of (2.8) these properties look as follows:



(2.10)



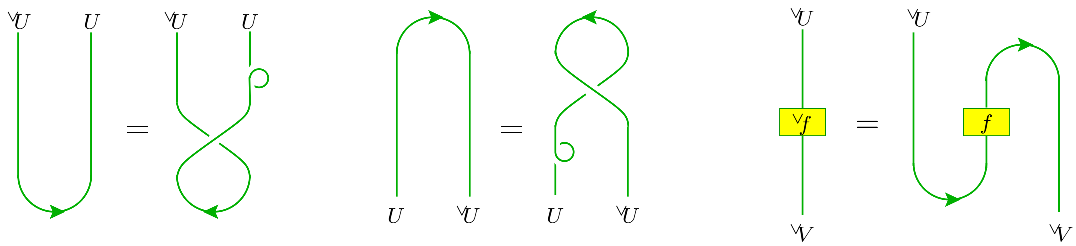
tensoriality of braiding



braiding and twist

(2.11)

In a ribbon category there is automatically also a left duality; it is defined on objects by ${}^\vee U := U^\vee$ and left duality morphisms $\tilde b_U$ and $\tilde d_U$; the latter, as well as left-dual morphisms ${}^\vee f$, are given by



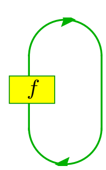
(2.12)

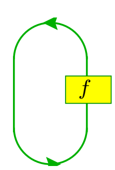
One can check that this left duality coincides with the right duality not only on objects, but also on morphisms, i.e. ${}^\vee f = f^\vee$; categories with a coinciding left and right duality are called *sovereign*. It follows e.g. that the double dual $(U^\vee)^\vee$ of an object U is isomorphic (though in general not equal) to U. In fact, a natural isomorphism between $(U^\vee)^\vee$ and U can be obtained with the help of the twist (see e.g. [39, Chapter 2.2]): For any object U we have a morphism $\delta_U := \psi_U^{-1} \circ \theta_U \in \mathrm{Hom}(U, (U^\vee)^\vee)$, with

$$\psi_U := (id_U \otimes d_{U^\vee}) \circ (id_U \otimes c^{-1}_{(U^\vee)^\vee, U^\vee}) \circ (b_U \otimes id_{(U^\vee)^\vee}) \in \mathrm{Hom}((U^\vee)^\vee, U)\,. \tag{2.13}$$

The properties of the twist θ_U are precisely such that these morphisms are tensorial, i.e. satisfy $\delta_{V\otimes W} = \delta_V \otimes \delta_W$, preserve the unit, $\delta_{\mathbf 1} = id_{\mathbf 1}$, and are compatible with the duality in the sense that $\delta_{U^\vee} = ((\delta_U)^\vee)^{-1}$.

Further, once we have two dualities, we can also define left and right traces of endomorphisms, via

$\mathrm{tr_r}(f) \;=$ 

$\mathrm{tr_l}(f) \;=$ 

(2.14)

Both traces are cyclic,

$$\mathrm{tr_{l,r}}(g \circ f) = \mathrm{tr_{l,r}}(f \circ g) \tag{2.15}$$

and obey

$$\mathrm{tr_{l,r}}(f \otimes g) = \mathrm{tr_{l,r}}(f)\, \mathrm{tr_{l,r}}(g)\,. \tag{2.16}$$

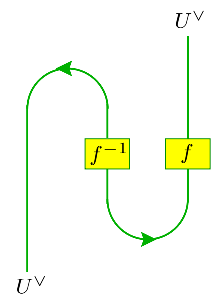
In the case at hand, where the left duality is constructed from the right duality by (2.12), the two notions of trace coincide; thus the category $\mathcal{C}$ is *spherical* [48]. The trace of the identity morphism is known as the *quantum dimension* of an object,

$$\dim(U) := \mathrm{tr}(id_U)\,. \tag{2.17}$$

The quantum dimension is additive under direct sums and multiplicative under tensor products.

For any self-dual object U of a sovereign tensor category and any isomorphism f in the space $\mathrm{Hom}(U,U^\vee)$ one introduces the endomorphism $\mathcal{V}_U \in \mathrm{Hom}(U^\vee,U^\vee)$ as ([36], see also [49, 50, 51])

$\mathcal{V}_U$ =



(2.18)

One can show [36] that $\mathcal{V}_U$ is in fact an automorphism, and that for simple self-dual U it does not depend on f and satisfies

$$\mathcal{V}_U = \nu_U \, id_{U^\vee} \tag{2.19}$$

with

$$\nu_U \in \{\pm 1\}\,. \tag{2.20}$$

The sign ν_U is called the *Frobenius–Schur indicator* of the object U. In agreement with the terminology in the representation theory of groups and Lie algebras, self-dual objects U with $\nu_U = 1$ are called real (or orthogonal), while those with $\nu_U = -1$ are called pseudo-real (or symplectic, or quaternionic).

■ Third, $\mathcal{C}$ is a *modular tensor category* (or, briefly, *modular category*), that is, a semisimple abelian ribbon category with ground field [5] $\mathbb{C}$ that has only a finite number of isomorphism classes of simple objects and a non-degenerate s-matrix.

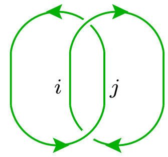
To explain the latter property, first note that in a modular tensor category the family of simple objects that appears in the definition of dominance (see formula (2.2)) can be chosen such that it contains precisely one object out of each isomorphism class of simple objects and hence in particular is finite, $|\mathcal{I}| < \infty$. Also, the dual $U^\vee$ of any member U of the family is isomorphic to another member $\bar{U}$, and as representative of the class of the tensor unit $\mathbf{1}$ we choose $\mathbf{1}$ itself. Note that $\bar{U} = U$ iff U is self-dual; in particular, $\bar{\mathbf{1}} = \mathbf{1}$. From now on we denote the elements of the family by U_i with $i \in \{0, 1, 2, \dots, |\mathcal{I}|-1\}$ and set $U_0 = \mathbf{1}$.

One then defines an $|\mathcal{I}| \times |\mathcal{I}|$-matrix $s = (s_{i,j})_{i,j\in\mathcal{I}}$ by

$$s_{i,j} := \mathrm{tr}(c_{U_i,U_j} c_{U_j,U_i})\,, \tag{2.21}$$

[5] In the original definition of a modular tensor category [12], semisimplicity is replaced by dominance and instead of abelianness also only a weaker property is imposed, nor is it required that the ground ring is $\mathbb{C}$. But in the context of rational conformal field theory modular tensor categories appear naturally in this more restricted form.

or pictorially,

$s_{i,j}$ = 

(2.22)

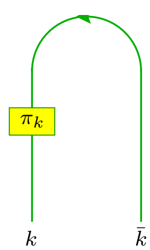
(Here and in the sequel we often simplify notation by writing the label i in place of U_i.) The final requirement to be imposed on $\mathcal{C}$ in order that it is a modular category is that this square matrix s is non-degenerate. This latter property provides in fact an explanation of the qualification 'modular': When combined with the other axioms, it implies that the matrices s and $t = \mathrm{diag}(\theta_{U_i})$ (when multiplied with overall constants that are expressible through the θ_{U_j} and the quantum dimensions $\dim(U_j)$) generate a projective representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$.

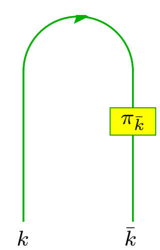
The tensor product of objects induces on the set of isomorphism classes of $\mathcal{C}$ the structure of a commutative and associative ring over the integers, called the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$. (Conversely, the tensor category $\mathcal{C}$ may be thought of as a categorification of the ring $K_0(\mathcal{C})$ [52].) A distinguished basis of this ring is given by the isomorphism classes of the objects U_i with $i \in \mathcal{I}$. In this basis, the structure constants are the non-negative integers $\dim \mathrm{Hom}(U_i \otimes U_j, U_k)$.

The mapping $U_k \mapsto \bar{U}_k$ is an involution on the finite set $\mathcal{I}$, which induces an involution $k \mapsto \bar{k}$ on the set of labels by $U_{\bar{k}} = \bar{U}_k$. With this convention, $U_{\bar{k}}^\vee$ is isomorphic to U_k, for every $k \in \mathcal{I}$; let us then fix an isomorphism

$$\pi_k \in \mathrm{Hom}(U_k, U_{\bar{k}}^\vee) \tag{2.23}$$

for each $k \in \mathcal{I}$, and for each pair $k, \bar{k}$ with $k \neq \bar{k}$ perform the choice in such a way that the number p_k defined by



= p_k 

(2.24)

is equal to 1. Using sovereignty of $\mathcal{C}$, one can then show that for these values of k we have consistently $p_{\bar{k}} = 1$, too. In contrast, for the self-dual objects $U_k = U_{\bar{k}}$ in the family, we are not free to choose the number p_k; rather, it follows directly from its definition in (2.24) that p_k coincides with the Frobenius–Schur indicator (2.20) of U_k:

$$p_k = \nu_k \qquad \text{for} \;\; U_k \cong U_k^\vee \,. \tag{2.25}$$

A particularly simple example of a modular tensor category, which can serve as a guide to the general theory, is the category $\mathcal{V}$ of finite-dimensional vector spaces over the complex numbers. The category $\mathcal{V}$ has a single isomorphism class of simple objects – the class of the one-dimensional vector space $\mathbb{C}$ – and has trivial twist and braiding. In conformal field theory, this category arises for meromorphic models, i.e. models with a single primary field, such as the E_8 WZW theory at level 1.

2.2 Fusing and braiding matrices

Let us now explain the meaning of the various properties of a modular category in the conformal field theory context. The (simple) objects of $\mathcal{C}$ are the (irreducible) representations of the chiral algebra $\mathfrak{V}$, and the morphisms of $\mathcal{C}$ are $\mathfrak{V}$-intertwiners. The tensor product is the (fusion) tensor product of $\mathfrak{V}$-representations, with the tensor unit given by the vacuum representation (identity field and its descendants). Thus the isomorphism classes of simple objects correspond to the primary chiral vertex operators, and the Grothendieck ring of $\mathcal{C}$ is the fusion ring of the conformal field theory.

The duality in $\mathcal{C}$ encodes the existence of conjugate $\mathfrak{V}$-representations, and the twist is determined by the fractional part of the conformal weight:

$$\theta_U = \exp(-2\pi\mathrm{i}\Delta_U)\, id_U \tag{2.26}$$

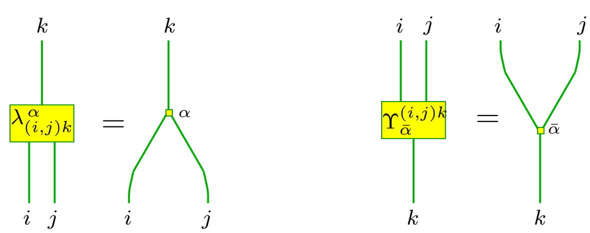
for simple objects U. The braiding of $\mathcal{C}$ accounts for the presence of braid group statistics (see e.g. [53, 54, 55, 56]) in two dimensions, and the matrix s coincides, up to normalisation, with the modular S-matrix of the CFT that implements the modular transformation $\tau \mapsto -\frac{1}{\tau}$ on the characters of primary fields:

$$s_{i,j} = S_{i,j}/S_{0,0}\,. \tag{2.27}$$

(Conversely, $S_{0,0}$ and thereby S is recovered from the data of the modular tensor category by requiring $S = S_{0,0}\, s$ to be unitary.) In terms of s, the quantum dimensions are

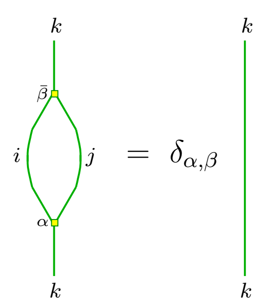
$$\dim(U_i) = s_{i,0} = S_{i,0}/S_{0,0}\,. \tag{2.28}$$

All the axioms of $\mathcal{C}$ can be understood as formalisations of properties of primary chiral vertex operators in rational CFT. Often such properties are presented in a form where explicit basis choices in the three-point coupling spaces have been made. To make contact with such a formulation we fix[6] once and for all bases $\{\lambda^{\alpha}_{(i,j)k}\}$ in the coupling spaces $\mathrm{Hom}(U_i\otimes U_j, U_k)$, as well as dual bases $\{\Upsilon^{(i,j)k}_{\bar\alpha}\}$ in $\mathrm{Hom}(U_k, U_i\otimes U_j)$. We depict the basis morphisms as follows:



(2.29)

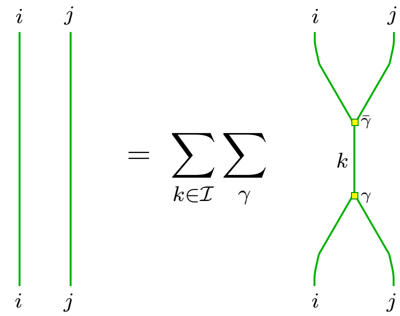
Duality of the bases means that



(2.30)

[6] Our strategy is to keep these bases as general as possible. More specific basis choices can be interesting for purposes different from ours, e.g. for an efficient numerical encoding of the defining data of a modular category as discussed in [57, 58].

By the dominance property of $\mathcal{C}$ we also have the completeness relation

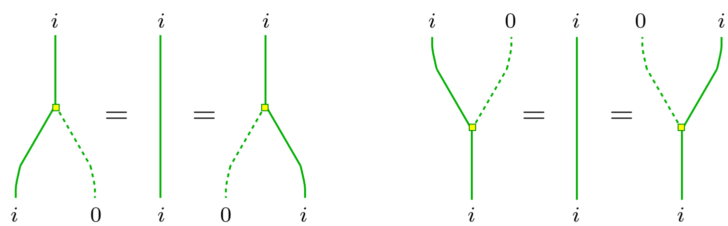


(2.31)

When all three labels i, j, k are generic, no basis in $\mathrm{Hom}(U_k, U_i \otimes U_j)$ or $\mathrm{Hom}(U_i \otimes U_j, U_k)$ is distinguished. In contrast, when one of the labels equals 0, i.e. when one of the simple objects involved is the tensor unit $\mathbf{1}$, then it is natural to make the choice

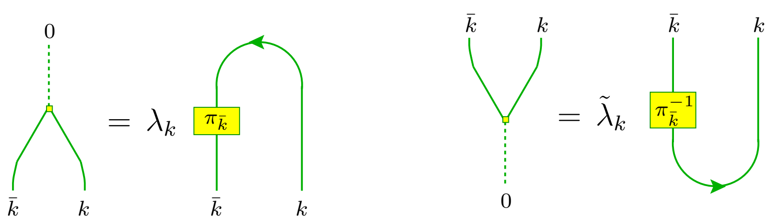
$$\lambda^{\circ}_{(i,0)i} = \lambda^{\circ}_{(0,i)i} = id_{U_i} = \Upsilon^{(i,0)i}_{\bar{\circ}} = \Upsilon^{(0,i)i}_{\bar{\circ}} , \tag{2.32}$$

(which is possible due to strictness of the tensor category $\mathcal{C}$). Here we have used the symbol $\circ$ in order to indicate that the coupling label can take only a single value. In the sequel, for notational simplicity we suppress such unique labels, both in the formulas and in the pictures. Thus e.g. the pictorial form of the relation (2.32) is



(2.33)

Since the spaces $\mathrm{Hom}(\bar{U}_k \otimes U_k, \mathbf{1})$ are one-dimensional, the morphisms $\lambda_{(\bar{k},k)0}$ and $\Upsilon^{(0,\bar{k})k}$ are proportional to the respective combinations of dualities and πs (as defined in (2.23)), i.e. there are numbers λ_k and $\tilde{\lambda}_k$ such that

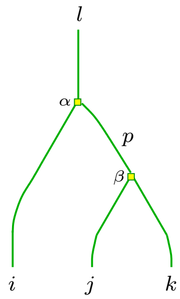


(2.34)

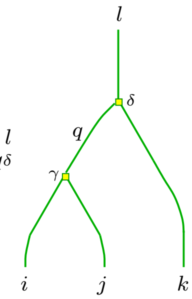
The normalisation condition (2.30) implies that the constants of proportionality are related by

$$\lambda_k \, \tilde{\lambda}_k = (\dim U_k)^{-1} \, . \tag{2.35}$$

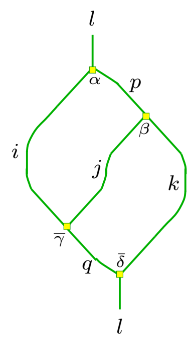
Recall now that $\mathcal{C}$ is a strict tensor category, i.e. that the tensor product of objects is strictly associative. Nevertheless, once we have chosen bases as above, there are two distinct distinguished bases for the morphism space $\mathrm{Hom}(U_i{\otimes}U_j{\otimes}U_k, U_l)$, corresponding to its two decompositions $\bigoplus_q \mathrm{Hom}(U_i{\otimes}U_j, U_q) \otimes \mathrm{Hom}(U_q{\otimes}U_k, U_l)$ and $\bigoplus_p \mathrm{Hom}(U_j{\otimes}U_k, U_p) \otimes \mathrm{Hom}(U_i{\otimes}U_p, U_l)$, respectively. The coefficients of the basis transformation between the two are known as the *fusing* matrices, F-matrices, or 6j-symbols of $\mathcal{C}$. We denote them as follows:



$$= \sum_q \sum_{\gamma,\delta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \qquad (2.36)$$



By composing with the morphism dual to the one on the right hand side, we arrive at the formula

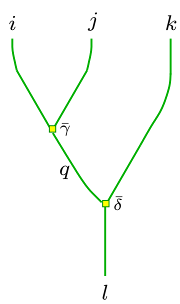


$$= \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \qquad (2.37)$$

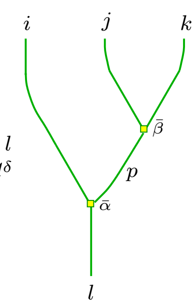
for the F-matrices. When any of the labels i, j, k equals 0, then the left hand side of formula (2.37) degenerates (if non-zero) to the left hand side of (2.30), leading to

$$\mathsf{F}^{(0\,j\,k)\,l}_{l\beta,j\delta} = \delta_{\beta,\delta}\,, \qquad \mathsf{F}^{(i\,0\,k)\,l}_{\alpha k,i\delta} = \delta_{\alpha,\delta}\,, \qquad \mathsf{F}^{(i\,j\,0)\,l}_{\alpha j,\gamma l} = \delta_{\alpha,\gamma}\,. \qquad (2.38)$$

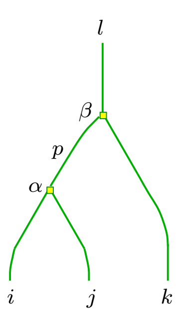
For the morphisms dual to those appearing in (2.36), there is an analogous relation



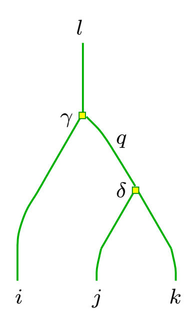
$$= \sum_p \sum_{\alpha,\beta} \mathsf{F}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \qquad (2.39)$$



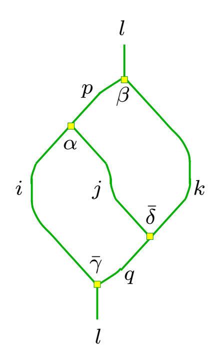
It is convenient to introduce a separate symbol G for the inverse of F. It is defined as



$$= \sum_{q,\gamma,\delta} \mathsf{G}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta}$$

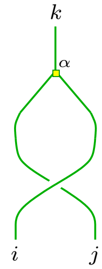


i.e.

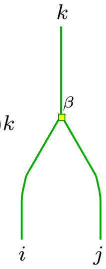


$$= \mathsf{G}^{(i\,j\,k)\,l}_{\alpha p\beta,\gamma q\delta} \qquad (2.40)$$

Combining the braiding morphisms with the basis choice (2.29) provides the *braiding* matrices R:



$$=: \sum_\beta \mathsf{R}^{(i\,j)k}_{\alpha\,\beta} \qquad (2.41)$$



The number that is obtained when the braiding $c_{i,j}$ ('over-braiding') is replaced by $c_{j,i}^{-1}$ ('under-braiding') is denoted by $\mathsf{R}^{-\,(i\,j)k}_{\alpha\,\beta}$, and instead of $\mathsf{R}^{(i\,j)k}_{\alpha\,\beta}$ one also often writes $\mathsf{R}^{+\,(i\,j)k}_{\alpha\,\beta}$. One easily checks that

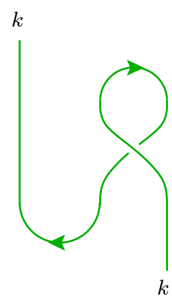
$$\sum_\beta \mathsf{R}^{(i\,j)k}_{\alpha\,\beta}\, \mathsf{R}^{-\,(j\,i)k}_{\beta\,\gamma} = \delta_{\alpha,\gamma} \qquad (2.42)$$

and, using the compatibility of twist and braiding and functoriality of the twist (see (2.11) and (2.10)), that

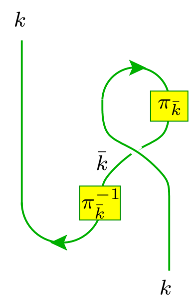
$$\sum_\beta \mathsf{R}^{(i\,j)k}_{\alpha\,\beta}\, \mathsf{R}^{(j\,i)k}_{\beta\,\gamma} = \frac{\theta_k}{\theta_i\,\theta_j}\, \delta_{\alpha,\gamma}\,. \qquad (2.43)$$

Here the complex number θ_k is defined by $\theta_{U_k} =: \theta_k\, id_{U_k}$, i.e. specifies the twist of the simple object U_k. We can express θ_k in terms of special matrix elements of F and R. To this end one rewrites the twist morphism of the object U_k as

$$=$$



$$= \qquad (2.44)$$



and then computes the constant by which the right hand side differs from id_{U_k} by first using the identity (2.34), then (2.41) and finally (2.37) (with $p\,{=}\,q\,{=}\,0$, $i\,{=}\,k$ and $j\,{=}\,\bar k$). The result is

$$\theta_k = \dim(U_k)\, \mathsf{F}^{(k\,\bar k\,k)\,k}_{0\,0}\, \mathsf{R}^{-\,(\bar k\,k)\,0}\,. \qquad (2.45)$$

The Frobenius–Schur indicator of a self-dual simple object U_k is encoded in the F matrix as well. To see this take formula (2.37) with $p\,{=}\,q\,{=}\,0$, $i\,{=}\,k\,{=}\,l$ and $j\,{=}\,\bar k$ for a not necessarily self-dual U_k. Then apply (2.34) and (2.35) to the left hand side and use (2.24) to cancel the morphisms π_k. The resulting relation reads

$$\dim(U_k)\, \mathsf{F}^{(k\,\bar k\,k)\,k}_{0\,0} = p_k\, \lambda_k\,/\lambda_{\bar k}\,. \qquad (2.46)$$

Finally specialise to $k = \bar k$ and employ (2.25) to arrive at

$$\nu_k = \dim(U_k)\, \mathsf{F}^{(k\,k\,k)\,k}_{0\,0}\,. \qquad (2.47)$$

Also note that combining (2.45) (or rather, the analogous equation obtained when using the inverse braiding) and (2.46) one has, for any simple U_k,

$$\mathsf{R}^{(\bar{k}\,k)\,0} = p_k\,\theta_k^{-1}\,\lambda_{\bar{k}}\,/\lambda_k\,. \tag{2.48}$$

For comparison with the literature, we note that our convention for the F- and R-matrices is related to the one of [47] by

$$\mathsf{F}_{p\,q}^{(j\,k\,l)\,i} \mathrel{\hat{=}} \mathsf{F}_{p,q}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \qquad \text{and} \qquad \mathsf{R}^{(i\,j)k} \mathrel{\hat{=}} \Omega_{ji}^{k} \tag{2.49}$$

(compare formula (2.4) and example 2.8 in [47]). Note that often also the composite quantities $\mathsf{B} \sim \Omega^{-}\mathsf{F}\,\Omega^{+}$ are used, see e.g. formula (3.3) in [47].

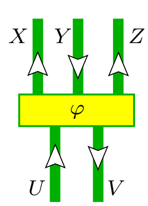
2.3 Three-manifolds and ribbon graphs

To determine a correlation function of a rational CFT, we specify it as a particular element in the relevant space of conformal blocks. A very convenient characterisation of conformal blocks is via ribbon graphs in three-manifolds. In this formulation, the coefficients in the expansion of a CFT correlator in terms of a chosen basis of conformal blocks are obtained as invariants of closed three-manifolds with embedded ribbon graphs. To explain this construction we need to introduce the concepts of a ribbon graph and of a three-dimensional topological field theory. (For more details see e.g. [59,60,61,62,63,64,65,66,12,26,39,67]; this quick introduction follows section 2 of [51].)

A *ribbon graph* consists of an oriented three-manifold M, possibly with boundaries, together with embedded ribbons and coupons. A *ribbon* is an oriented rectangle, say $[-1/10, 1/10] \times [0, 1]$, together with an orientation for its *core* $\{0\} \times [0, 1]$. The two subsets $[-1/10, 1/10] \times \{0\}$ and $[-1/10, 1/10] \times \{1\}$ are the *ends* of the ribbon. A *coupon* is an oriented rectangle with two preferred opposite edges, called top and bottom. The embeddings of ribbons and coupons into M are demanded to be injective. A ribbon minus its ends does not intersect any other ribbon, nor any coupon, nor the boundary of M. A coupon does not intersect any other coupon nor the boundary of M. The ends of a ribbon must either lie on one of the preferred edges of some coupon or on ∂M. For ribbons ending on a coupon, the orientation of the ribbon and of the coupon must agree.

Choosing an orientation for the ribbons and coupons is equivalent to choosing a preferred side – henceforth called the 'white' side – which in the drawings will usually face the reader. The opposite ('black') side is drawn in a darker shade, as has already been done in figure (2.9) above. We use open arrows to indicate the orientation of the ribbon's core.

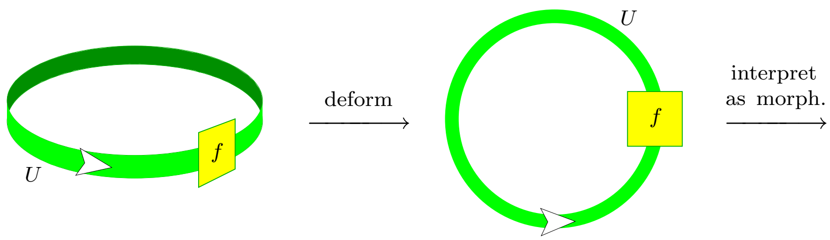
Each constituent ribbon of a ribbon graph is labelled (sometimes called 'colored') by a (not necessarily simple) object of the modular category $\mathcal{C}$, and each coupon is labelled by a morphism of $\mathcal{C}$. Which space the morphism belongs to depends on the ribbons ending on the coupon. If a ribbon labelled by U is 'incoming', the relevant object is U when the ribbon is attached to the bottom of the coupon, and $U^\vee$ when it is attached to its top. If it is 'outgoing', then the convention for the object is the other way round. As an illustration, the coupon in



(2.50)

is labelled by an element $\varphi \in \mathrm{Hom}(U \otimes V^\vee, X \otimes Y^\vee \otimes Z)$.

Consider now a ribbon graph in S^3. We can assign an element in $\mathrm{Hom}(\mathbf{1},\mathbf{1})$, i.e. a complex number, to it as follows: Regard S^3 as $\mathbb{R}^3 \cup \{\infty\}$, with $\mathbb{R}^3$ parametrised by Cartesian coordinates (x, y, z). Deform the ribbon graph such that all its coupons are rectangles in the x-y-plane, with white sides facing upwards, and such that the top and bottom edges are parallel to the x-axis, with the bottom edge having a smaller y-coordinate than the top edge. Deform the ribbons to lie in a small neighbourhood of the x-y-plane. In short, the ribbon graph must be arranged in such a manner that the bends, twists and crossings of all ribbons can be expressed as dualities, twists and braidings as appearing in (2.8) and (2.9). The element in $\mathrm{Hom}(\mathbf{1},\mathbf{1})$ is then obtained by reading the graph from $y = -\infty$ to $y = +\infty$ and interpreting it as a concatenation of morphisms in $\mathcal{C}$. Let us display an example:



$$d_U \circ (id_{U^\vee} \otimes f) \circ \tilde{b}_U \tag{2.51}$$

One of the non-trivial results following from the defining relations of a modular category is that different ways to translate a ribbon graph into a morphism give rise to one and the same value for the ribbon graph. In other words, the value for the ribbon graph is invariant under the various local moves that transform those different descriptions into each other.

All pictures in sections 2.5–4.5 below directly stand for morphisms in $\mathcal{C}$. The first genuine ribbon graphs will occur in section 5.1. We will usually simplify the pictures involving ribbon graphs by replacing all ribbons by lines. When doing so, it is understood that the ribbon lies in the plane of the paper, with the white side facing up (this convention is known as 'blackboard framing'). Also note that, strictly speaking, the definition of a ribbon graph given above forbids annulus-shaped ribbons. Whenever such a ribbon occurs below it is understood to be of the form displayed in (2.51), with f the identity morphism in $\mathrm{Hom}(U,U)$.

2.4 Topological field theory

So far we have used the modular category $\mathcal{C}$ only to assign numbers to ribbon graphs in S^3; for this purpose it is actually sufficient that $\mathcal{C}$ is a ribbon category. That $\mathcal{C}$ is even modular implies the highly non-trivial result that it gives rise to a full three-dimensional topological field theory. By definition, a three-dimensional TFT is a pair $(Z, \mathcal{H})$ of assignments that associate algebraic structures to geometric data – extended surfaces and cobordisms – and satisfy various properties, to be be outlined below.

By an *extended surface* we mean an oriented closed compact two-manifold X with a finite number of disjoint oriented arcs (the remnants, at the topological level, of the local coordinates that one must choose around the insertion points of the world sheet) labelled by pairs (U, ε) with $U \in \mathcal{O}bj(\mathcal{C})$ and $\varepsilon \in \{\pm 1\}$, and with a lagrangian subspace $L(\mathrm{X})$ of the first homology group $H_1(\mathrm{X}, \mathbb{R})$. The *opposite* $-\mathrm{X}$ of an extended surface is obtained from X by reversing the orientation of X, reversing the orientation of all arcs and replacing ε by $-\varepsilon$. To make explicit the insertions $U, V, \cdots$ of an extended surface we will sometimes write $(U, V, \cdots; \mathrm{X})$ instead of X. This notation assumes that all signs ε are $+1$, but does not encode the positions and orientations of the arcs; when the sign ε for some object is -1, then we write $(U,-)$ instead of U.

The first datum, $\mathcal{H}$, of the TFT assigns to each extended surface X a finite-dimensional vector space $\mathcal{H}(\mathrm{X})$. This space is constructed from appropriate tensor products of morphism spaces of $\mathcal{C}$; one has

$$\mathcal{H}(\emptyset) = \mathbb{C}\,. \tag{2.52}$$

The CFT interpretation of $\mathcal{H}(\mathrm{X})$ is as the space of conformal blocks on X. Thus we have, for example, $\dim\mathcal{H}(k;S^2) = \delta_{k,0}$, in agreement with the fact that the space of one-point blocks on the sphere is zero-dimensional except for the insertion of a vacuum. Similarly, $\dim\mathcal{H}(k,(\ell,-);S^2) = \delta_{k,\ell}$, i.e. the two-point blocks on the sphere vanish unless the two fields are dual to each other. More generally,

$$\dim\mathcal{H}(i,j,(k,-);S^2) = N_{ij}^{\ k} \tag{2.53}$$

(recall that $N_{ij}^{\ k} = \dim\mathrm{Hom}(U_i\otimes U_j, U_k) = \dim\mathrm{Hom}(U_i\otimes U_j\otimes U_k^\vee, \mathbf{1})$). Each homeomorphism $f\colon \mathrm{X}\to\mathrm{Y}$ of extended surfaces induces a linear map $f_\natural\colon \mathcal{H}(\mathrm{X})\to\mathcal{H}(\mathrm{Y})$.

Consider now a manifold M with ribbon graph whose boundary can be written as a disjoint union $\partial M = \partial_1 M \sqcup \partial_2 M$. Turn $\partial_1 M$ and $\partial_2 M$ into extended surfaces by taking as arcs the ends of ribbons, with orientation induced by the ribbons. In addition fix a Lagrangian subspace (this is part of the defining data of an extended surface; it is not determined by M and the embedded ribbon graph). When a ribbon ending on ∂M is labelled by U, then the corresponding arc is labelled by $(U,+1)$ if the core of the ribbon points away from the surface, and by $(U,-1)$ otherwise. Denote the extended surface $\partial_2 M$ by $\partial_+ M$ and $-\partial_1 M$ by $\partial_- M$. Then the triple $(M,\partial_- M,\partial_+ M)$ is called a *cobordism* from $\partial_- M$ to $\partial_+ M$.

The second datum, Z, of the TFT assigns a linear map

$$Z(M,\partial_- M,\partial_+ M):\quad \mathcal{H}(\partial_- M)\to\mathcal{H}(\partial_+ M) \tag{2.54}$$

to every cobordism. Let us mention two of its properties. The first concerns the normalisation of Z. Let X be an extended surface and M the manifold $\mathrm{X}\times[0,1]$ with embedded ribbon graph given by straight ribbons connecting the arcs in $\mathrm{X}\times\{0\}$ to the arcs in $\mathrm{X}\times\{1\}$, with cores oriented from 0 to 1. Then

$$Z(M,\mathrm{X},\mathrm{X}) = id_{\mathcal{H}(\mathrm{X})}\,. \tag{2.55}$$

The second property is functoriality. Let M_1 and M_2 be two three-manifolds with ribbon graphs and let $f\colon \partial_+ M_1\to\partial_- M_2$ be a homeomorphism of extended surfaces, and let M be the manifold obtained from glueing M_1 to M_2 using f. Then

$$Z(M,\partial_- M_1,\partial_+ M_2) = \kappa^m\, Z(M_2,\partial_- M_2,\partial_+ M_2)\circ f_\natural\circ Z(M_1,\partial_- M_1,\partial_+ M_1)\,, \tag{2.56}$$

where m is an integer (see the review in [51] for details) and $\kappa = S_{0,0}\sum_{j\in\mathcal{I}}\theta_j^{-1}\dim(U_j)^2$, which is called the *charge* of the modular category $\mathcal{C}$.

In our application we will not need the properties of Z in their most general form. But we will use the following special cases and consequences: For the manifolds $S^2\times S^1$ and S^3 without ribbon graph we have

$$Z(S^2{\times}S^1;\emptyset;\emptyset) = 1 \qquad\text{and}\qquad Z(S^3;\emptyset;\emptyset) = S_{0,0}\,. \tag{2.57}$$

More generally, for S^3 with embedded ribbon graph, the number assigned by Z is $S_{0,0}$ times the number obtained by translating the ribbon graph to a morphism of $\mathcal{C}$ as described in the previous section.

Functoriality of Z implies that the invariant of a ribbon graph in any closed three-manifold can be related to an invariant in S^3 by the use of surgery along links. This will be used in section 5.2 and 5.7 to relate invariants of $S^2 \times S^1$ and S^3.

We will also make intensive use of a trace formula that is obtained as follows: For X an extended surface, consider the three-manifold $N = \mathrm{X} \times [0,1]$ with embedded ribbon graph such that $\partial_- N = \partial_+ N = \mathrm{X}$ as extended surfaces. Let M be the closed three-manifold with ribbon graph obtained from N by identifying $(x,0)$ with $(x,1)$ for all $x \in \mathrm{X}$. Then

$$Z(M,\emptyset,\emptyset) = \mathrm{tr}_{\mathcal{H}(\mathrm{X})}\, Z(N,\mathrm{X},\mathrm{X})\,. \tag{2.58}$$

This trace formula, too, is a consequence of the functoriality of Z.

2.5 The case ${N_{ij}}^k \in \{0,1\}$

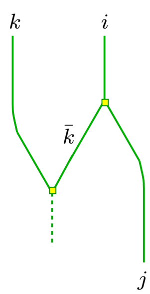
In this section we specialise the general treatment above to the case that the dimensions ${N_{ij}}^k$ of all coupling spaces $\mathrm{Hom}(U_i \otimes U_j, U_k)$ are either 0 or 1. This greatly simplifies both notation and calculation. It may be thought of as a 'meta-example'; in particular, it encompasses both concrete examples that we study in this paper.

The main notational simplification is that the multiplicity indices disappear, so that the fusing and braiding matrices take the form

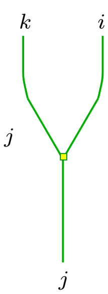
$$\mathsf{F}_{p\,q}^{\,(i\,j\,k)\,\ell} \qquad \text{and} \qquad \mathsf{R}^{\,(i\,j)k}\,, \tag{2.59}$$

respectively. (But we must still choose bases in the morphism spaces, and the form of the fusing and braiding matrices does depend on this choice.) The quantum dimensions $\dim(U_k)$ (which in a modular category are positive real numbers) can be obtained from F and R, by taking the absolute value of (2.45) (since θ_k is a phase). The twist eigenvalues θ_k follow from (2.45) as well, and finally the Frobenius–Schur indicators are given by (2.47). (In practice, it is often easier to obtain $\dim(U_k)$ and θ_k by some other means, though.)

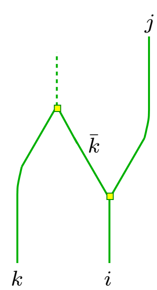
To reconstruct the S-matrix from F and R, as well as for later use, the following identities prove to be useful:



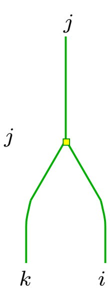
$$= \mathsf{F}_{i\,0}^{\,(k\,\bar{k}\,j)\,j}$$



and



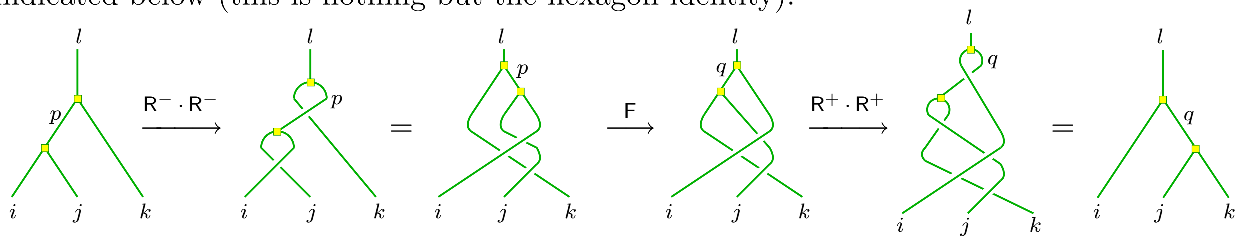
$$= \mathsf{G}_{0\,i}^{\,(k\,\bar{k}\,j)\,j} \tag{2.60}$$



as well as

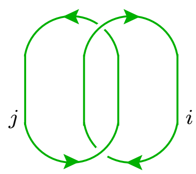
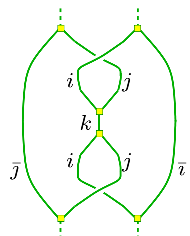
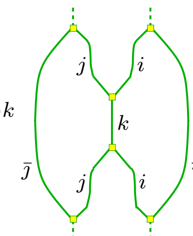
$$\mathsf{G}_{p\,q}^{\,(i\,j\,k)\,\ell} = \frac{\mathsf{R}^{\,(j\,k)q}\,\mathsf{R}^{\,(i\,q)\ell}}{\mathsf{R}^{\,(i\,j)p}\,\mathsf{R}^{\,(p\,k)\ell}}\,\mathsf{F}_{p\,q}^{\,(k\,j\,i)\,\ell}\,. \tag{2.61}$$

The first equation, for instance, follows by composing with the three-point coupling dual to the right hand side and noticing that the left hand side thereby becomes a special case $(q=0)$ of the graph on the left hand side of (2.37) that gives the general F-matrix element. The last equation relating the F-matrix to its inverse can be seen by following the sequence of moves indicated below (this is nothing but the hexagon identity):



(2.62)

The S-matrix can then be expressed through the F- and R-matrices by the moves in the following figure:

$$\frac{1}{\dim(U_i)\dim(U_j)}$$

$$= \sum_{k\in\mathcal{I}}$$

$$= \sum_{k\in\mathcal{I}} \mathsf{R}^{(i\,j)k}\,\mathsf{R}^{(j\,i)k}$$


(2.63)

Using now the two relations in (2.60) as well as (2.43), this results in

$$S_{i,j} = S_{0,0}\dim(U_i)\dim(U_j) \sum_{k\in\mathcal{I}} \frac{\theta_k}{\theta_i\,\theta_j}\, \mathsf{G}^{(\bar\jmath\, j\, i)\, i}_{0\, k}\, \mathsf{F}^{(\bar\jmath\, j\, i)\, i}_{k\, 0}\,. \tag{2.64}$$

Recall that we chose the basis in the spaces of three-point couplings in such a way that an F-matrix is equal to one (see (2.38)) when one of the 'ingoing' objects is the tensor unit $\mathbf{1}$. It is convenient to have a similar behavior when the 'outgoing' object is $\mathbf{1}$. The following lemma implies that we can always make choices such that $\mathsf{F}^{(i\,j\,k)\,0}_{\bar\imath\,\bar k} = p_i p_j p_k$:

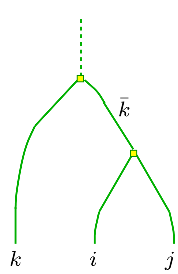
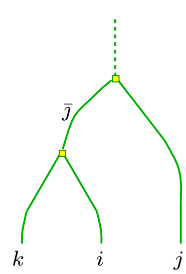
Lemma 2.1 :
If $\dim \mathrm{Hom}(X\otimes Y, Z) \in \{0,1\}$ for all simple objects $X, Y, Z \in \mathcal{O}bj(\mathcal{C})$, then there is a choice of basis in the spaces of three-point couplings such that

$$\mathsf{F}^{(i\,j\,k)\,\ell}_{p\,q} = 1 \tag{2.65}$$

whenever the F-matrix element is allowed to be non-zero by the fusion rules and one or more of the i, j, k are 0, and that

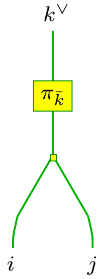
$$F_{ijk} \equiv \mathsf{F}^{(i\,j\,k)\,0}_{\bar\imath\,\bar k} = p_i\, p_j\, p_k\,. \tag{2.66}$$

Proof:
Recall the conventions (2.33) and (2.34) that we have made already. For the sake of this proof, we will specialise the choice of basis further such that $\lambda_k = \lambda_{\bar k}$. By definition of F we have

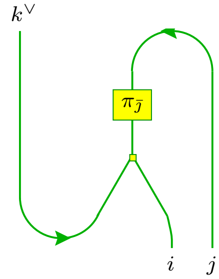

$$= \quad F_{kij}$$


(2.67)

Then we have the relation



$$= \; p_k \, \frac{\lambda_j}{\lambda_k} \, F_{kij}$$

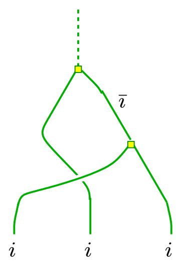


(2.68)

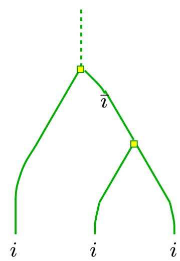
which when iterated results in

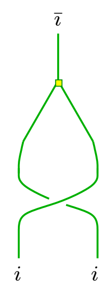
$$F_{kij} \, F_{jki} \, F_{ijk} = p_i \, p_j \, p_k \,. \tag{2.69}$$

When the three coupling spaces that form an orbit upon iteration, i.e. $\mathrm{Hom}(i{\otimes}j,\bar k)$, $\mathrm{Hom}(k{\otimes}i,\bar\jmath)$ and $\mathrm{Hom}(j{\otimes}k,\bar\imath)$, are mutually distinct, then we can link the choice of basis in the latter two to the choice for the first in such a manner that $F_{kij} = F_{jki} = p_i p_j p_k$. Relation (2.69) then implies that $F_{ijk} = p_i p_j p_k$ as well.
That the three coupling spaces are not mutually distinct can happen only if $i = j = k$. Let us abbreviate from here on F_{iii} by F_i. Define the two numbers B_i and R_i via

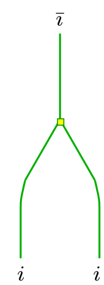


$$= \; B_i$$





$$= \; R_i$$



(2.70)

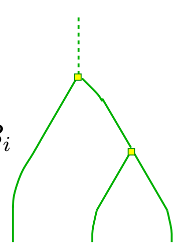
The number B_i exists because the space $\mathrm{Hom}(U_i \otimes U_i \otimes U_i, \mathbf{1})$ has dimension one and R_i is just $\mathsf{R}^{(i\,i)\bar\imath}$, compare formula (2.41).

Next note the relations

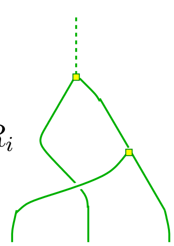
$$F_i F_i F_i = p_i \,, \qquad F_i = p_i \theta_i B_i R_i \,, \qquad B_i B_i = 1/\theta_i \,, \qquad R_i R_i = 1/\theta_i \,. \tag{2.71}$$

The second relation, for instance, is obtained by the moves (we omit the obvious labels)

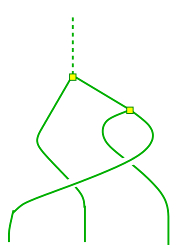
$$R_i B_i$$



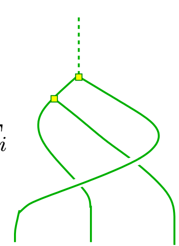
$$= \; R_i$$



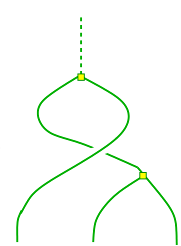
$$=$$



$$= \; F_i$$



$$= \; F_i$$



(2.72)

together with formula (2.48) for $\mathsf{R}^{(i\,\bar\imath)0}$. Using the relations (2.71), it follows that

$$F_i F_i = (p_i \theta_i B_i R_i)^2 = 1 \,, \tag{2.73}$$

and thus $p_i = F_i F_i F_i = F_i$, which proves the lemma. ✓

2.5.1 Example: Free boson

The first example is the modular category associated to the chiral data of a free boson field compactified on a circle of rational radius squared. The generic chiral algebra of this theory is $\hat{\mathfrak{u}}(1)$. If the compactification radius R fulfills (in the normalisation we choose) $R^2 = p/q$ with p, q coprime, then $\hat{\mathfrak{u}}(1)$ can be extended by a pair of Virasoro primary fields of weight $N = pq$.

The modular category for the Moore–Seiberg data of this theory can be presented as follows (see e.g. [68]). There is a simple object U_k for each integer k. The simple objects U_k and U_{k+2N} are isomorphic, so that the category has $2N$ isomorphism classes of simple objects; we label them as $\mathcal{I} = \{0, 1, \dots, 2N{-}1\}$. (Note that we should not ignore these isomorphisms and pretend that U_k and U_{k+2N} are equal. This is, for instance, in line with the fact that the operator product of two fields of $\mathfrak{u}(1)$ charge $j/\sqrt{2N}$ and $k/\sqrt{2N}$ in the standard range, i.e. $j, k \in \mathcal{I}$, does not contain any field with charge in the standard range when $j{+}k \geq 2N$.) The conformal weight of the primary field labelled by $k \in \mathcal{I}$, whose non-integral part determines the twist of the object U_k, is

$$\Delta(k) = \begin{cases} \frac{1}{4N}\, k^2 & \text{for } k \leq N\,, \\ \frac{1}{4N}\,(2N{-}k)^2 & \text{for } k > N\,. \end{cases} \tag{2.74}$$

The representation k of the extended chiral algebra contains $\hat{\mathfrak{u}}(1)$-representations of charges $q(k) = (k{+}2Nm)/\sqrt{2N}$ for every integer m (in the convention $\Delta(k) = (q(k))^2/2$). From the conformal weights we read off that

$$\theta_k = \mathrm{e}^{-2\pi\mathrm{i}\Delta(k)} = \mathrm{e}^{-\pi\mathrm{i}k^2/2N}\,. \tag{2.75}$$

To give the braiding and fusing matrices, it is useful to introduce the function $[\,\cdot\,]\colon \mathbb{Z} \to \mathcal{I}$ that associates to n the element of $\mathcal{I}$ with which it coincides modulo $2N$, i.e. $[n] = n - 2N\,\sigma(n)$ with $\sigma(n) \equiv (n{-}[n])/2N$ the unique integer such that $n - 2N\sigma(n)$ is in $\mathcal{I}$. Then the fusion rules, which furnish the abelian group $\mathbb{Z}_{2N}$, read

$$[i] \star [j] = [i{+}j]\,, \tag{2.76}$$

and with our conventions for the coupling spaces the non-zero braiding and fusing matrices are given by

$$\begin{aligned} &\mathsf{F}^{(i\,j\,k)\,[i+j+k]}_{\;[j+k]\,[i+j]} = (-1)^{(i+k+1)\{j\sigma(i+j+k)\,+\,(j+k)(\sigma(i+j)+\sigma(j+k))\}}\,, \\ &\mathsf{R}^{(k\,\ell)[k+\ell]} = (-1)^{(k+\ell)\sigma(k+\ell)}\,\mathrm{e}^{-\pi\mathrm{i}k\ell/(2N)}\,. \end{aligned} \tag{2.77}$$

Two points should be noted: First, had we chosen another way to represent the isomorphism classes of simple objects, these formulas would look different. In particular, they depend on our specific choice for the set $\mathcal{I}$, and indeed are not invariant under the shift $n \mapsto n{+}2N$. Second, by redefining the bases of the coupling spaces $\mathrm{Hom}(j{\otimes}k, [j{+}k])$ by a factor $(-1)^{j(k+1)\sigma(j+k)}$, the F's could be made to look much simpler, namely $\mathsf{F}^{(i\,j\,k)\,[i+j+k]}_{\;[j+k]\,[i+j]} = (-1)^{\sigma(j+k)\,i}$, which is the form used in [69]. This gauge does not, however, fulfill $\mathsf{F}^{(i\,j\,k)\,0}_{\;\bar\imath\,\bar k} = p_i p_j p_k$ (which we use in simplifying the explicit expressions in the examples treated), whereas (2.77) does.

2.5.2 Example: $\mathfrak{sl}(2)$ WZW models

The chiral algebra of the $\mathfrak{sl}(2)$ WZW model at level k is the affine Lie algebra $\mathfrak{sl}(2)_k$. It has $k{+}1$ isomorphism classes of integrable highest weight modules, which we label by their highest weights (twice the spin): (0), (1), ... , (k). The corresponding conformal weights read

$$\Delta(n) = \frac{n\,(n+2)}{4\,(k+2)}\,, \tag{2.78}$$

and the quantum dimensions are

$$\dim\,(n) = S_{0,n}\,/\,S_{0,0} = \sin\,(\tfrac{\pi(n+1)}{k+2})\,/\,\sin\,(\tfrac{\pi}{k+2})\,. \tag{2.79}$$

The fusion rules read

$$(m) \star (n) = \sum_{p=|m-n|}^{\min(m+n,2k-m-n)} (p)\,, \tag{2.80}$$

where p is increased in steps of 2.

The concrete expressions for the braiding and fusing matrices depend on the chosen normalisation of the chiral vertex operators. We will use the form in which they naturally arise as quantum group 6j-symbols, as given e.g. in [70, 71]:

$$\mathsf{R}^{(r\,s)t} = (-1)^{(r+s-t)/2}\,\mathrm{e}^{-\mathrm{i}\pi(\Delta(t)-\Delta(r)-\Delta(s))}\,, \qquad \mathsf{F}^{(r\,s\,t)\,u}_{p\,q} = \begin{Bmatrix} t/2 & s/2 & p/2 \\ r/2 & u/2 & q/2 \end{Bmatrix}_q \tag{2.81}$$

with

$$\begin{aligned} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q := (-1)^{a+b-c-d-2e}([2e{+}1]\,[2f{+}1])^{1/2}\Delta(a,b,e)\,\Delta(a,c,f)\,\Delta(c,e,d)\,\Delta(d,b,f) \\ \times \textstyle\sum_z (-1)^z\,[z{+}1]!\,\Big(\,[z{-}a{-}b{-}e]!\,[z{-}a{-}c{-}f]!\,[z{-}b{-}d{-}f]!\,[z{-}d{-}c{-}e]! \\ \times\,[a{+}b{+}c{+}d{-}z]!\,[a{+}d{+}e{+}f{-}z]!\,[b{+}c{+}e{+}f{-}z]!\,\Big)^{-1} \end{aligned} \tag{2.82}$$

and

$$\Delta(a,b,c) := \sqrt{[-a{+}b{+}c]!\,[a{-}b{+}c]!\,[a{+}b{-}c]!\,/\,[a{+}b{+}c{+}1]!}\,. \tag{2.83}$$

The symbols $[n]$ and $[n]!$ stand for q-numbers and q-factorials, respectively, i.e.

$$[n] = \frac{\sin\,(\frac{\pi n}{k+2})}{\sin\,(\frac{\pi}{k+2})}\,, \qquad [n]! = \prod_{m=1}^{n}\,[m]\,, \qquad [0]! = 1\,. \tag{2.84}$$

(The fusion rules ensure that the numbers that appear as arguments of $[\,\cdot\,]$ are always integral.) The range of the summation is such that the arguments are non-negative, i.e. z runs over all integers (in steps of 1) from $\max(a{+}b{+}e, a{+}c{+}f, b{+}d{+}f, c{+}d{+}e)$ to $\min(a{+}b{+}c{+}d, a{+}d{+}e{+}f, b{+}c{+}e{+}f)$.

It is worth pointing out that, while these expressions solve the pentagon and hexagon identities, to find the normalisation of WZW conformal blocks such that their transformation behaviour under fusion and braiding is described by the matrices (2.81) is a non-trivial task. In the sequel, however, we will mainly be interested in partition functions; these are independent of the normalisation of the chiral vertex operators.

3 Frobenius objects and algebras of open string states

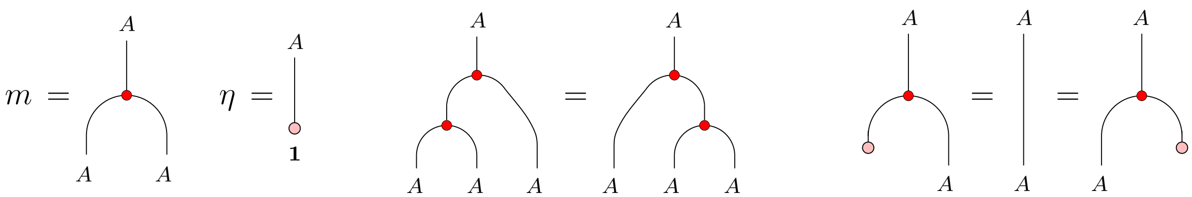
3.1 Algebra objects

In the construction of correlation functions a central role will be played by objects with additional structure in the modular category of the chiral CFT, namely so-called algebra objects. It is quite natural to study such extra structures for objects of a tensor category; for some category theoretic background, see [25, chapter VI] and [72, chapters 2,3]. In short, an algebra in a tensor category $\mathcal{C}$ is an object $A \in \mathcal{O}bj(\mathcal{C})$ together with a product, i.e. a morphism between $A \otimes A$ and A, that is associative and has a unit. More concretely:

Definition 3.1:
An *algebra object*, or simply an *algebra*, in a tensor category $\mathcal{C}$ is a triple (A, m, η), where A is an object of $\mathcal{C}$, $m \in \mathrm{Hom}(A \otimes A, A)$ and $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, such that the multiplication morphism m and the unit morphism η fulfill

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m) \qquad \text{and} \qquad m \circ (\eta \otimes id_A) = id_A = m \circ (id_A \otimes \eta)\,. \tag{3.1}$$

Our pictorial notation for m and η and the resulting pictures expressing formulas (3.1) are as follows:



(3.2)

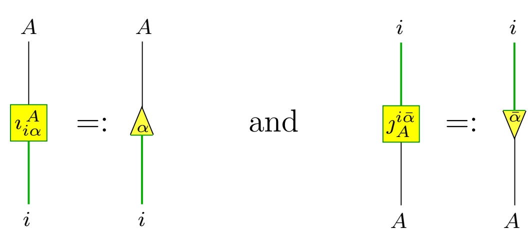
Later on, we will often suppress the label A on the algebra lines when this is unambiguous, e.g. when the presence of morphisms depicted by small circles – products or units – indicates that we are dealing with morphisms involving only A (and possibly $\mathbf{1}$).

To call A an algebra is appropriate because in the particular case that $\mathcal{C}$ is the category of vector spaces over $\mathbb{C}$ (or some other field), the prescription reduces to the conventional notion of an algebra. In every tensor category the tensor unit $\mathbf{1}$ provides a trivial example of an algebra; its product is $m = id_{\mathbf{1}} \equiv id_{\mathbf{1}\otimes\mathbf{1}} = id_{\mathbf{1}} \otimes id_{\mathbf{1}}$, and its unit is $\eta = id_{\mathbf{1}}$. A class of less trivial examples, present in any tensor category with duality, is given by objects of the form $A = U \otimes U^\vee$; in these cases one can take

$$m = id_U \otimes d_U \otimes id_{U^\vee} \qquad \text{and} \qquad \eta = b_U\,. \tag{3.3}$$

However, it turns out that many interesting algebras are not of this special form.

Consider now an algebra A in a semisimple tensor category $\mathcal{C}$. The object A is a finite direct sum of simple subobjects U_i. We fix once and for all bases $\{\imath_{i\alpha}^A\}$ in the embedding spaces $\mathrm{Hom}(U_i, A)$, as well as dual bases $\{\jmath_A^{i\bar\alpha}\}$ in $\mathrm{Hom}(A, U_i)$. We draw these morphisms as



(3.4)

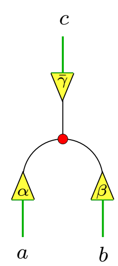
In accordance with the notation (2.3), the dimensions of these coupling spaces will be abbreviated by

$$\langle i,A\rangle := \dim \mathrm{Hom}(U_i, A) = \dim \mathrm{Hom}(A, U_i)\,. \tag{3.5}$$

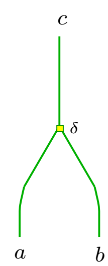
Thus as an object we have $A \cong \bigoplus_{k\in\mathcal{I}} U_k^{\oplus\langle k,A\rangle}$. The unit morphism η is non-zero, and hence

$$\langle 0,A\rangle = \dim \mathrm{Hom}(\mathbf{1}, A) \geq 1\,. \tag{3.6}$$

We may express the product m with respect to the bases $\{\imath_{i\alpha}^{A}\}$ of $\mathrm{Hom}(U_i, A)$ and $\{\jmath_{A}^{i\bar\alpha}\}$ of $\mathrm{Hom}(A, U_i)$, for $i = a, b, c$, as compared to the basis $\{\lambda_{(a,b)c}^{\delta}\}$ of $\mathrm{Hom}(U_a \otimes U_b, U_c)$ that we introduced in (2.29). This way we characterise m by the collection of numbers $m_{a\alpha,b\beta}^{c\gamma;\,\delta}$ that are defined as



$$= \sum_{\delta=1}^{N_{ab}{}^{c}} m_{a\alpha,b\beta}^{c\gamma;\,\delta} \tag{3.7}$$



Upon use of formula (2.36) one finds that in terms of these numbers $m_{a\alpha,b\beta}^{c\gamma;\,\delta}$, the associativity property of m is expressed as

$$\sum_{\varphi=1}^{\langle f,A\rangle} m_{a\alpha,b\beta}^{f\varphi;\,\rho}\, m_{f\varphi,c\gamma}^{d\delta;\,\sigma} = \sum_{e\in\mathcal{I}} \sum_{\varepsilon=1}^{\langle e,A\rangle} \sum_{\varrho=1}^{N_{bc}{}^{e}} \sum_{\tau=1}^{N_{ae}{}^{d}} m_{b\beta,c\gamma}^{e\varepsilon;\,\varrho}\, m_{a\alpha,e\varepsilon}^{d\delta;\,\tau}\, \mathsf{F}_{\tau e\varrho,\rho f\sigma}^{\,(a\,b\,c)\,d}\,. \tag{3.8}$$

This shows that the concept of associativity we are dealing with is most natural indeed: When expressing the associativity property in terms of bases, it involves precisely the F-matrices, a behavior that is familiar from the representation theory of algebras over $\mathbb{C}$, like e.g. universal enveloping algebras of simple Lie algebras.

3.2 From boundary conditions to algebras

We will now argue that each boundary condition of a rational CFT determines an algebra in the modular category for that CFT. To explain this relationship, let us start on the CFT side, i.e. with some boundary condition M.

For definiteness take the world sheet geometry to be the upper half plane. Then the boundary is the real line $\mathbb{R}$. Let M be a boundary condition that preserves the chiral algebra $\mathfrak{V}$. We do not assume, however, that $\mathfrak{V}$ is maximally extended, so M may break some of the symmetries of the model; but we do assume that the theory is still rational with respect to $\mathfrak{V}$.

The space $\mathcal{H}$ of states that corresponds to boundary fields living on a boundary segment with boundary condition M – first introduced in [73] – is often referred to as the 'state space on the upper half plane with boundary condition M' or, in string terminology, as the space of open string states for open strings stretching from one D-brane to itself. $\mathcal{H}$ is organised in modules of one copy of the chiral algebra $\mathfrak{V}$, rather than in modules of $\mathfrak{V}\otimes\mathfrak{V}$ as for the bulk:

$$\mathcal{H} = \bigoplus_{a\in\mathcal{I}} n^a\, U_a\,. \tag{3.9}$$

The sum runs over (representatives for the isomorphism classes of) irreducible highest weight modules U_a of $\mathfrak{V}$, and $n^a \in \mathbb{Z}_{\geq 0}$ specifies their multiplicities. In the CFTs we consider, the representation theory of $\mathfrak{V}$ gives rise to a modular category $\mathcal{C}$ whose simple objects are the irreducible $\mathfrak{V}$-modules. The notation U_a for these modules is chosen so as to conform with our convention for the representatives for isomorphism classes of simple objects in $\mathcal{C}$, see section 2.1 above.

There is a one-to-one correspondence between states in $\mathcal{H}$ and fields that live on the boundary M. Let us denote primary boundary fields (i.e. those that correspond to highest weight states) on M as $\Psi_{a\alpha}(x)$. Here a labels the $\mathfrak{V}$-representation, while α is a multiplicity index that runs from 1 to n^a. Since Ψ is a boundary field, its argument x takes values in $\mathbb{R}$ only.

In simple cases, for instance in minimal models, the operator product expansion of boundary fields takes the form

$$\Psi_{a\alpha}(x)\,\Psi_{b\beta}(y) = \sum_{c,\gamma} C_{a\alpha,b\beta}^{\;c\gamma}\,(x{-}y)^{\Delta_c-\Delta_a-\Delta_b}\,\big[\,\Psi_{c\gamma}(y) + \text{terms with descendants}\,\big]\,. \tag{3.10}$$

This defines the boundary operator product coefficients (or structure constants) $C_{a\alpha,b\beta}^{\;c\gamma}$ of primary boundary fields on M.

In the general case two related additional features must be taken into account: First, there may be more than one independent way for representations a and b to fuse to c. Second, for a given coupling of a and b to c, the primary field of c does not necessarily appear [7] on the right hand side of (3.10). The dimension of the space of couplings is the fusion rule $N_{ab}^{\;c}$.

To obtain a formulation of the OPE that covers the generic situation we regard a coupling of a and b to c as a prescription on how to associate to every vector v in the highest weight module U_a of the chiral algebra a linear map from U_b to U_c. A basis in the space of couplings is then a collection of $N_{ab}^{\;c}$ maps ${V_{ab}}^{c,\delta}\colon U_a \to \mathrm{Hom}(U_b,U_c)[[z,z^{-1}]]\cdot z^{\Delta_c-\Delta_a-\Delta_b}$, $\delta = 1,2,\ldots,N_{ab}^{\;c}$; the basis elements are known as chiral vertex operators. [8] Further, we denote by $\{v_d^D\}_D$ an orthonormal basis of L_0-eigenvectors in U_d, with $v_d^0 \equiv v_d$ a highest weight vector, and by $\Psi_{d\alpha}^D$ the corresponding descendant field of the primary $\Psi_{d\alpha} \equiv \Psi_{d\alpha}^0$. Then the OPE reads

$$\begin{aligned}\Psi_{a\alpha}(x)\,\Psi_{b\beta}(y) \;=& \sum_{c\in\mathcal{I}}\sum_{\gamma=1}^{n^c}\sum_{\delta=1}^{N_{ab}^{\;c}} C_{a\alpha,b\beta}^{\;c\gamma;\,\delta}\,(x{-}y)^{\Delta_c-\Delta_a-\Delta_b} \\ & \sum_{C} \langle v_c^C|{V_{ab}}^{c,\delta}(v_a;z{=}1)|v_b\rangle\,(x{-}y)^{\Delta(v_c^C)-\Delta_c}\,\Psi_{c\gamma}^C(y)\,.\end{aligned} \tag{3.11}$$

Note that here we consider only special cases of boundary operator products, since we are interested in a single boundary condition M. To be fully general, we must also deal with boundary fields $\Psi_{d\alpha}^{MM'}$ that change the boundary condition, from M to M', and hence operator products $\Psi_{a\alpha}^{MM'}(x)\,\Psi_{b\beta}^{M'M''}(y)$ expanded in boundary fields $\Psi_{c\gamma}^{MM''}(y)$. Correspondingly, the operator product coefficients acquire three more labels M,M',M''. As we will discuss in section 4.4, boundary changing fields play a natural role in the categorical setup as well; they occur in the CFT interpretation of representations of algebra objects.

[7] In fact, with a suitable basis choice the primary field appears for at most one of the couplings.

[8] $\mathrm{Hom}(U_b,U_c)[[z,z^{-1}]]$ denotes the space of Laurent series with values in the space $\mathrm{Hom}(U_b,U_c)$. To make the concept of chiral vertex operators precise, one should work in the vertex operator algebra setting, where z is a formal variable, so that one is dealing with formal Laurent series.

The sewing constraint [74] that arises from the factorisation of the correlator of four boundary fields on a disk with boundary condition M then looks as follows:

$$\sum_{\varphi=1}^{n_f} C_{a\alpha,b\beta}^{\,f\varphi;\,\rho}\, C_{f\varphi,c\gamma}^{\,d\delta;\,\sigma} = \sum_{e\in\mathcal{I}} \sum_{\varepsilon=1}^{n_e} \sum_{\varrho=1}^{N_{bc}^{\ e}} \sum_{\tau=1}^{N_{ae}^{\ d}} C_{b\beta,c\gamma}^{\,e\varepsilon;\,\varrho}\, C_{a\alpha,e\varepsilon}^{\,d\delta;\,\tau}\, \mathsf{F}_{\tau e\varrho,\rho f\sigma}^{\,(a\,b\,c)\,d}\,. \tag{3.12}$$

It proves to be convenient to describe the boundary structure constants with the help of the concepts of the *category $\mathcal{C}_A$ of A-modules* and of a *module category*, which will be described in section 4.1 below. They correspond to the generalised 6j-symbols ${}^{(1)}\!F$ (defined via formula (4.9)) of $\mathcal{C}_A$ regarded as a module category, and the relation (3.12) is nothing but the corresponding generalisation of the pentagon identity. The identification between the boundary structure constants C and the quantities ${}^{(1)}\!F$ also appears in the formalism used in [14], where it is obtained via the relation with weak Hopf algebras; our formulas (3.11) and (3.12) correspond to (4.11) and (4.29) of [14] for a single boundary condition. In the Cardy case the ${}^{(1)}\!F$ specialise to the ordinary F-matrices (this was first observed for the A-series of Virasoro minimal models in [75] and established in general in [14, 76]); the sewing constraint (3.12) is then just the pentagon identity for F.

From the data provided by the decomposition (3.9) of the state space and by formula (3.11) for the structure constants we can obtain an algebra A in $\mathcal{C}$, which we call the algebra of open string states. This is achieved as follows. As an object in $\mathcal{C}$, the algebra of open string states coincides with $\mathcal{H}$ as given in formula (3.9), i.e.

$$A \cong \bigoplus_{a\in\mathcal{I}} n^a\, U_a\,. \tag{3.13}$$

Now let us choose a basis in $\mathrm{Hom}(U_a, A)$ as described in section 3.1, and define a multiplication m on A in this basis via (3.7), by demanding that the coefficients $m_{a\alpha,b\beta}^{\,c\gamma;\,\delta}$ are just the structure constants:

$$m_{a\alpha,b\beta}^{\,c\gamma;\,\delta} := C_{a\alpha,b\beta}^{\,c\gamma;\,\delta}\,. \tag{3.14}$$

With this assignment the consistency condition (3.12) fulfilled by the structure constants of the boundary fields on M coincides with the associativity condition (3.8) for the algebra A. The unit of A is the identity field on M. We conclude that indeed the fields living on a boundary with boundary condition M that preserves $\mathfrak{V}$ provide us with an algebra object A in $\mathcal{C}$.

The algebra objects arising from boundary conditions enjoy an additional important property which derives from the non-degeneracy of the two-point correlation functions. Non-degeneracy means that for any field Ψ on the M-boundary there exists at least one [9] Ψ' such that

$$\langle\, \Psi(x)\, \Psi'(y)\, \rangle \neq 0\,. \tag{3.15}$$

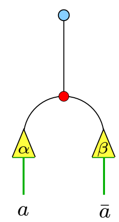
If there were a field in the theory that had zero two-point function with all other fields, it would decouple from the theory, i.e. every correlator involving that field would vanish.

To evaluate the two-point functions one can use the OPE (3.11) together with the one-point functions $\langle\Psi(x)\rangle$ of boundary fields. The latter can be non-zero only for boundary fields

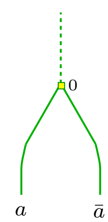
[9] We cannot formulate this condition in terms of primary fields alone, for the same reason that we had to adopt the more complicated form (3.11) of the OPE. E.g. in WZW models generically it takes a (horizontal) descendant to get a non-zero two-point function with a primary field.

of weight zero, but (as is natural in view of the connection of our results with those in two-dimensional lattice TFT) we allow for the possibility that there can be more of those than just the identity field. A superposition of two elementary boundaries, for example, has at least two fields of weight zero. In the case at hand, the one-point functions are determined once we know them for the primary boundary fields $\Psi_{a\alpha}$ on M.

In the category theoretic setting, the collection of one-point functions of boundary fields on M will give rise to a morphism $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$. The non-degeneracy (3.15) translates into the non-degeneracy of the composition $\varepsilon \circ m$. That is, the matrix $G(a)_{\alpha\beta}$ defined by



$$= \quad G(a)_{\alpha\beta}$$



(3.16)

is invertible as a matrix in the multiplicity labels α, β.

In fact, the one-point functions $\langle \Psi_{a\alpha}(x) \rangle$ of primary boundary fields on the upper half plane (or equivalently, on the unit disk) are themselves of a special form. First of all, by conformal invariance they can be non-zero only for $a = 0$. A more interesting result follows from the sewing constraint for the boundary one-point functions on an annulus.

Consider a cylinder of length L and circumference T with boundary condition M at both ends. On one of the boundaries insert a boundary field $\Psi_{0\alpha}$ of weight zero. The cylinder can be conformally mapped to an annulus of inner radius q and outer radius 1, and alternatively to a half-annulus (with the two half-circular boundaries identified) in the upper half plane, with the two ends stretching from -1 to $-\tilde{q}$ and from $\tilde{q}$ to 1, where

$$q := \exp(-2\pi L/T) \qquad \text{and} \qquad \tilde{q} := \exp(-\pi T/L)\,. \tag{3.17}$$

This results in the equation

$$\langle M(\Psi_{0\alpha})|\, q^{L_0+\bar{L}_0-c/12}\, |M\rangle = \mathrm{tr}_{\mathcal{H}}(\, \tilde{q}^{\,L_0-c/24}\, \Psi_{0\alpha}\,)\,, \tag{3.18}$$

where $|M(\Psi_{0\alpha})\rangle$ denotes the boundary state for the M-boundary with an insertion of the boundary field $\Psi_{0\alpha}$ and $|M\rangle$ the boundary state without field insertion. On the right hand side of (3.18), the boundary field $\Psi_{0\alpha}$ is interpreted as an operator on the boundary state space $\mathcal{H}$.

Let us now assume that the CFT under consideration does not possess any state of negative conformal weight (as is in particular the case for all unitary theories) and that it has a unique vacuum (i.e., state $|0\rangle$ of weight zero) in the *bulk*, which we take to be normalised as $\langle 0|0\rangle = 1$. Then in the limit of infinite length L only the vacuum propagates on the left hand side of (3.18), so that

$$\langle M(\Psi_{0\alpha})|\, q^{L_0+\bar{L}_0-c/12}\, |M\rangle = q^{-c/12}\,(1+O(q^{\kappa}))\,\langle M(\Psi_{0\alpha})|0\rangle\,\langle 0|M\rangle \qquad \text{for} \quad L\to\infty \tag{3.19}$$

with $\kappa > 0$. Finding the $L\to\infty$ limit of the right hand side of (3.18) requires slightly more work. In accordance with the notation introduced in (3.11), let $|\Psi^D_{d\delta}\rangle$ denote a basis of $\mathcal{H}$, and

$\langle\Psi^D_{d\delta}|$ a dual basis, i.e. $\langle\Psi^C_{c\gamma}|\Psi^D_{d\delta}\rangle = \delta_{c,d}\delta_{\gamma,\delta}\delta_{C,D}$. Then

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}\, \tilde q^{\,L_0-c/24}\, \Psi_{0\alpha} &= \sum_{d\in\mathcal{I}}\sum_{\delta=1}^{n^d}\sum_{D\ \text{desc.}} \langle\Psi^D_{d\delta}|\, \tilde q^{\,L_0-c/24}\, \Psi_{0\alpha}\,|\Psi^D_{d\delta}\rangle \\
&= \textstyle\sum_{d,\delta}\langle\Psi_{d\delta}|\, \Psi_{0\alpha}\,|\Psi_{d\delta}\rangle\, \chi_d(\tilde q) = \sum_{d,\delta}\sum_{c\in\mathcal{I}} C_{0\alpha,d\delta}^{\;d\delta}\, S_{d,c}\, \chi_c(q^2) \\
&= q^{-c/12}\,(1+O(q^\kappa)) \textstyle\sum_{d,\delta} C_{0\alpha,d\delta}^{\;d\delta}\, S_{d,0} \qquad \text{for}\ \ L\to\infty\,.
\end{aligned} \tag{3.20}$$

In the first step, the trace is written out in terms of the bases. The crucial step is the second one. Since $\Psi_{0\alpha}$ is a field of weight zero, it behaves just like the identity field as far as commutation with elements of the chiral algebra is concerned, and this in turn implies that

$$\langle\Psi^D_{d\delta}|\, \tilde q^{\,L_0-c/24}\, \Psi_{0\alpha}\,|\Psi^D_{d\delta}\rangle = \tilde q^{\,\Delta(\Psi^D_{d\delta})-c/24}\, \langle\Psi_{d\delta}|\, \Psi_{0\alpha}\,|\Psi_{d\delta}\rangle\,, \tag{3.21}$$

so that the sum over descendants D gives rise to the character χ_d. The third step involves a modular transformation of this character, upon which finally the limit as $L\to\infty$ can be taken, leaving only the contribution of the vacuum character. [10]

Comparing the results (3.19) and (3.20) we conclude that the one-point functions of weight zero boundary fields on the unit disk are given by

$$\langle\Psi_{0\alpha}\rangle = \mathrm{const} \sum_{d\in\mathcal{I}}\sum_{\delta=1}^{n^d} S_{d,0}\, C_{0\alpha,d\delta}^{\;d\delta}\,. \tag{3.22}$$

Note that here the sum runs over all primary boundary fields on the M-boundary, not only over those of zero weight. When translated back into the language of algebra objects, this relation reads (using the basis (3.4))

$$\varepsilon\circ\imath^A_{0\alpha} = \mathrm{const}' \sum_{d\in\mathcal{I}}\sum_{\delta=1}^{\langle d,A\rangle} m_{0\alpha,d\delta}^{\;d\delta}\, \dim(U_d)\,. \tag{3.23}$$

The additional properties (3.16) and (3.23) of the multiplication m mean that – provided that the CFT under consideration has a unique bulk vacuum – the algebra object A can be turned into what will be called a symmetric special Frobenius algebra. It is the aim of the next section to explain these notions. Before entering that discussion, let us make some short remarks on the situation that $Z_{00}>1$ for the torus partition function, taking $Z_{00}=2$ as an example. In this case there exist two distinct bulk fields of weight zero. All correlators are independent of the insertion points of these weight zero fields, thus they form a topological subsector of the theory. In particular, their operator product algebra is commutative. As a consequence there is a projector basis $\{P_1,P_2\}$ for the weight zero fields, in which their operator products take the form $P_1P_1=P_1$, $P_2P_2=P_2$ and $P_1P_2=0$. The identity field is then given by $\mathbf{1}=P_1+P_2$. The projector fields P_1,P_2 act on the space of fields via the OPE, and thereby decompose the space of fields into eigenspaces. Let us denote by $\phi_1(z,\bar z)$ bulk fields for which $P_1\phi_1(z,\bar z)=\phi_1(z,\bar z)$, and analogously for ϕ_2. Mixed correlators then vanish:

$$\begin{aligned}
\langle\, \phi_1(z,\bar z)\, \phi_2(w,\bar w)\ \cdots\rangle &= \langle\, (P_1\phi_1(z,\bar z))\, \phi_2(w,\bar w)\ \cdots\rangle \\
&= \langle\, \phi_1(z,\bar z)\, (P_1\phi_2(w,\bar w))\ \cdots\rangle = 0\,.
\end{aligned} \tag{3.24}$$

[10] Here we assume that of the representations U_i of the chiral algebra $\mathfrak{V}$, the vacuum representation U_0 is the only one which contains a state of weight zero.

We conclude that a CFT with $Z_{00} = 2$ should be interpreted as a superposition of two CFTs each of which has $Z_{00} = 1$. By a *superposition* of two conformal field theories, CFT_1 and CFT_2, with the same central charge we mean the theory in which fields are pairs $\Phi = (\phi_1, \phi_2)$ of fields in the two constituent CFTs and correlation functions are sums

$$\langle\, \Phi(z,\bar z)\, \Phi(w,\bar w)\, \cdots \rangle = \langle\, \phi_1(z,\bar z)\, \phi_1(w,\bar w)\, \cdots \rangle_1 + \langle\, \phi_2(z,\bar z)\, \phi_2(w,\bar w)\, \cdots \rangle_2 \,. \qquad (3.25)$$

Requiring that a CFT cannot be written as a superposition of this type has a counterpart in the underlying algebra object, which in this case must be indecomposable in the sense of definition 9(ii) of [13].

3.3 Frobenius algebras

The construction of correlation functions that will be discussed in sections 5.3 and 5.8 (and further in forthcoming papers) uses a symmetric special Frobenius algebra object as an input. In the present section we explain the qualifiers special, symmetric, and Frobenius, and we show that algebras with these properties are natural from the CFT point of view. This is the content of theorem 3.6 below.

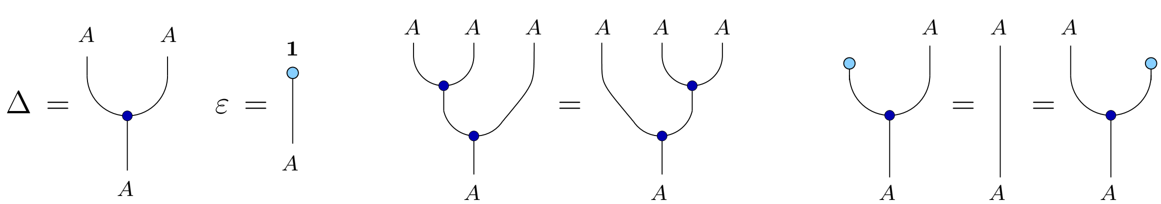
From a computational point of view, the construction of such an algebra object requires the solution of the boundary factorisation constraint (3.12) for boundary fields living on *one single* boundary – rather than a simultaneous solution of all constraints involving all boundary conditions as well as boundary changing fields, which form a much larger system. It turns out that the construction of the algebra is the *only* place where a *non-linear* constraint ever must be solved. Finding the other boundary conditions and structure constants then amounts to solving systems of linear equations only.

The notions of co-algebra (which is the notion dual to that of an algebra), Frobenius algebra, special and symmetric given below are straightforward extensions to the category setting of the corresponding concepts in algebras over $\mathbb{C}$, see e.g. [23].

Definition 3.2:
A *co-algebra* A in a tensor category $\mathcal{C}$ is an object with a coassociative coproduct Δ and a counit ε, i.e. morphisms $\Delta \in \mathrm{Hom}(A, A{\otimes}A)$ and $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ that satisfy

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta \qquad \text{and} \qquad (\varepsilon \otimes id_A) \circ \Delta = id_A = (id_A \otimes \varepsilon) \circ \Delta \,. \qquad (3.26)$$

The pictures for these morphisms and their properties are obtained by turning upside-down those for an algebra:



(3.27)

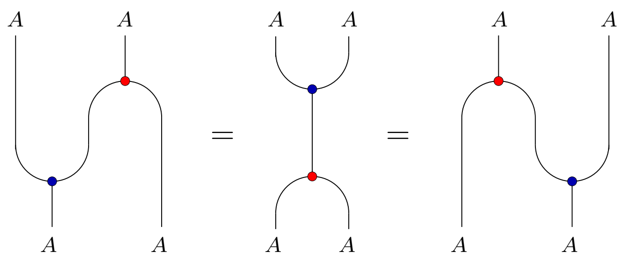
We will be interested in objects that possess both an algebra and a co-algebra structure and for which these structures are interrelated in a special way. This is encoded in the definition of a Frobenius algebra (see [77, 78] and references therein). Note that provided the tensor category has a braiding (a condition that is not needed in the Frobenius case) one may combine the algebra and co-algebra structure also into the one of a bialgebra; this amounts to a different compatibility condition between product and coproduct. Bi- or Hopf algebras in tensor categories do not occur in our discussion of boundary conditions, though they play an important role in other contexts (see e.g. [79, 80, 81, 82, 83]), and we refrain from describing those structures here.

Definition 3.3:

A *Frobenius algebra* in a tensor category $\mathcal{C}$ is an object that is both an algebra and a co-algebra and for which the product and coproduct are related by

$$(id_A \otimes m) \circ (\Delta \otimes id_A) = \Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta)\,. \tag{3.28}$$

In pictures,



(3.29)

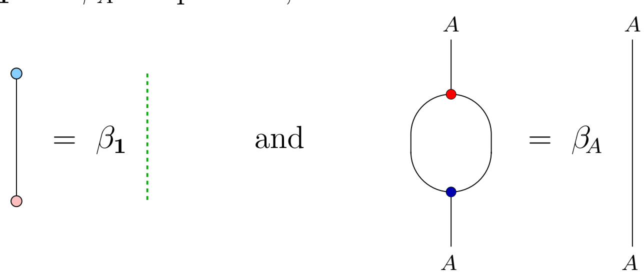
To be able to discuss additional properties of the algebras that come from boundary conditions, let us further introduce the following notions.

Definition 3.4:

(i) A *special* algebra in a tensor category $\mathcal{C}$ is an object that is both an algebra and a co-algebra such that

$$\varepsilon \circ \eta = \beta_{\mathbf{1}}\, id_{\mathbf{1}} \qquad \text{and} \qquad m \circ \Delta = \beta_A\, id_A \tag{3.30}$$

for non-zero numbers $\beta_{\mathbf{1}}$ and β_A. In pictures,

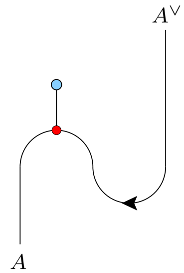


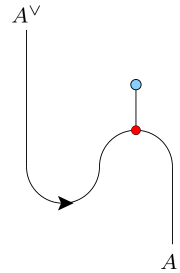
(3.31)

(ii) A *symmetric* algebra in a sovereign tensor category $\mathcal{C}$ is an algebra object (A, m, η) together with a morphism $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ such that the two morphisms $\Phi_1, \Phi_2 \in \mathrm{Hom}(A, A^\vee)$ defined as

$$\Phi_1 := [(\varepsilon \circ m) \otimes id_{A^\vee}] \circ (id_A \otimes b_A) \qquad \text{and} \qquad \Phi_2 := [id_{A^\vee} \otimes (\varepsilon \circ m)] \circ (\tilde{b}_A \otimes id_A) \tag{3.32}$$

are equal. In pictorial notation the morphisms Φ_1 and Φ_2 are given by

$\Phi_1 \;=$ 

$\Phi_2 \;=$ 

(3.33)

(iii) A *commutative* algebra in a braided tensor category $\mathcal{C}$ is an algebra object (A, m, η) for which $m \circ c_{A,A} = m$.

(iv) A *haploid* algebra in a tensor category is an algebra object A for which $\dim \mathrm{Hom}(\mathbf{1}, A) = 1$.

Remark 3.5:

(i) In the study of algebras over $\mathbb{C}$ [23], symmetric algebras constitute an interesting subclass of Frobenius algebras. They contain for instance all group algebras. Also, the property of being symmetric is considerably weaker than being commutative (abelian). For algebras in modular tensor categories, commutativity (with respect to the braiding c) and triviality of the twist (i.e. $\theta_A = id_A$) together imply that A is symmetric.

(ii) Algebras possessing the same representation theory are called Morita equivalent (see e.g. [84, 85, 22, 13, 78]). It turns out that neither haploidity nor commutativity are invariant under Morita equivalence. As will be discussed in more detail elsewhere, the algebra objects $U \otimes U^\vee$ described below definition 3.1 turn out to be Morita equivalent for different (not necessarily simple) U. But while for $U = \mathbf{1}$ the resulting algebra is certainly haploid and commutative, for a general object U this need not be the case.

(iii) In practice, haploidity is a quite useful property, because a haploid algebra is automatically symmetric, as follows from corollary 3.10 below. Furthermore it can be shown [13, section 3.3] that every Morita class of algebras that cannot be decomposed non-trivially in a direct sum has at least one haploid representative. (For the notion of direct sum of algebras see proposition 3.21 below.)

(iv) The algebras A of our interest are not necessarily (braided-) commutative. However, commutativity is still an interesting property, because in CFT terms it immediately allows for an interpretation of A as an extension of the chiral algebra (see section 5.5 below). Note, however, that according to point (ii) above, a non-commutative algebra can still correspond to an extension. Commutative algebras A in a braided tensor category have also been studied under the name 'quantum subgroups' [86, 11]. This terminology has its origin in the fact [87] that commutative algebras in the representation category of a group G are in one-to-one correspondence with the algebras of functions on homogeneous spaces G/H, and thereby with subgroups H of G.

In section 3.2 above we have seen that algebra objects are a natural structure to consider from the CFT point of view, as they can be obtained from boundary conditions. In the rest of this section we show that:

Theorem 3.6 :
(i) If an algebra can be endowed with the structure of a symmetric special Frobenius algebra, then this structure is unique up to a normalisation constant $\xi \in \mathbb{C}^\times$.
(ii) Every algebra coming from a boundary condition that preserves the chiral algebra $\mathfrak{V}$ of a (rational, unitary) CFT with unique bulk vacuum can be endowed with the structure of a symmetric special Frobenius algebra.

Note that the results of this paper rely only on the existence of a symmetric special Frobenius algebra. Part (ii) of theorem 3.6 guarantees the existence of such an algebra in a large class of theories. It does not exclude, however, the existence of such an algebra in theories not belonging to this class, in particular in certain non-unitary theories (e.g. non-unitary minimal models).

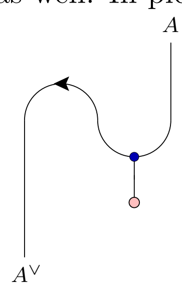
The proof of the theorem will require several lemmata.

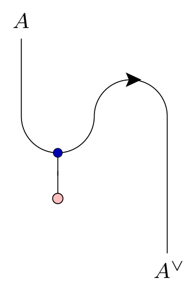
Lemma 3.7 :
Let (A, m, η) be an algebra and let $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$. Then the following holds.
(i) If there exists a Frobenius structure on A with counit ε, then it is unique.
(ii) There exists a Frobenius structure on A if and only if Φ_1 as defined in (3.32) is invertible.

Proof:
(i) We must show that the coproduct Δ is unique. If there exists any coproduct Δ satisfying the Frobenius property, then Φ_1 as defined in (3.32) is invertible, with (left- and right-) inverse Φ_1^{-1} given by

$$\Phi_1^{-1} = [d_A \otimes id_A] \circ [id_{A^\vee} \otimes (\Delta \circ \eta)] \,. \tag{3.34}$$

Analogously, Φ_2 is invertible as well. In pictures, the inverses are

$\Phi_1^{-1} =$ 

$\Phi_2^{-1} =$ 

(3.35)

Moreover, application of the Frobenius property also shows that we can express, conversely, the coproduct through the product and the morphism (3.34):

$$\Delta = (id_A \otimes m) \circ (id_A \otimes \Phi_1^{-1} \otimes id_A) \circ (b_A \otimes id_A) \tag{3.36}$$

(a picture can be found below in (3.42)). Now for given m and ε, Φ_1 is uniquely determined by (3.32). Since in addition the morphism Φ_1^{-1} is uniquely determined by its property of being inverse to Φ_1, it follows that Δ is unique as well.

(ii) We have already seen in the proof of part (i) that A being Frobenius implies that the morphism Φ_1 is invertible, with inverse given by (3.34). Conversely, let us now assume invertibility of Φ_1. One quickly checks that Φ_2 as given in (3.32) can be expressed through Φ_1 as[11]

$$\Phi_2 = [id_{A^\vee} \otimes d_A] \circ [id_{A^\vee} \otimes \Phi_1 \otimes id_A] \circ [\tilde{b}_A \otimes id_A] \,. \tag{3.37}$$

[11] Note that Φ_2 does not, in general, coincide with the morphism $\Phi_1^\vee$ that is dual to Φ_1. Indeed, $\Phi_1^\vee$ is an element of $\mathrm{Hom}(A^{\vee\vee}, A^\vee)$, and while $A^{\vee\vee}$ is certainly isomorphic to A there is no reason why the two objects should be equal.

Consequently it is invertible as well, with inverse given by

$$\Phi_2^{-1} = [id_A \otimes \tilde{d}_A] \circ [id_A \otimes \Phi_1^{-1} \otimes id_{A^\vee}] \circ [b_A \otimes id_{A^\vee}] \,. \tag{3.38}$$

It follows that we can define a candidate coproduct Δ by formula (3.36), and yet another candidate Δ' by the corresponding formula with Φ_2^{-1}, i.e.

$$\Delta' := (m \otimes id_A) \circ (id_A \otimes \Phi_2^{-1} \otimes id_A) \circ (id_A \otimes \tilde{b}_A) \,. \tag{3.39}$$

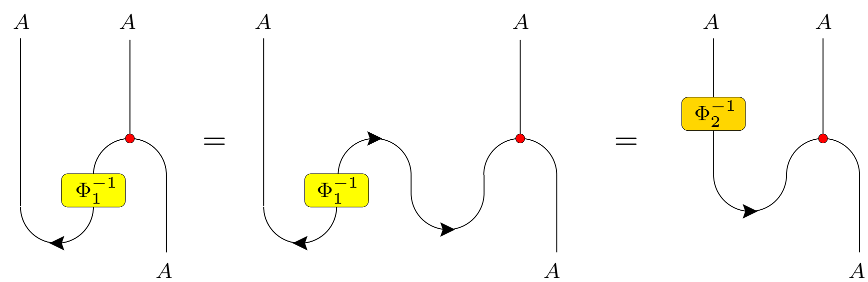
Note that because of the relation (3.38), we can also write

$$\Delta = (id_A \otimes m) \circ (\Phi_2^{-1} \otimes id_A \otimes id_A) \circ (\tilde{b}_A \otimes id_A) \tag{3.40}$$

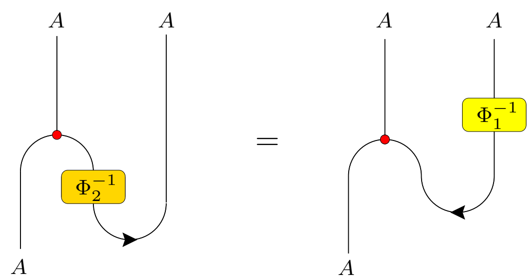
and

$$\Delta' = (m \otimes id_A) \circ (id_A \otimes id_A \otimes \Phi_1^{-1}) \circ (id_A \otimes b_A) \,, \tag{3.41}$$

respectively. Pictorially,

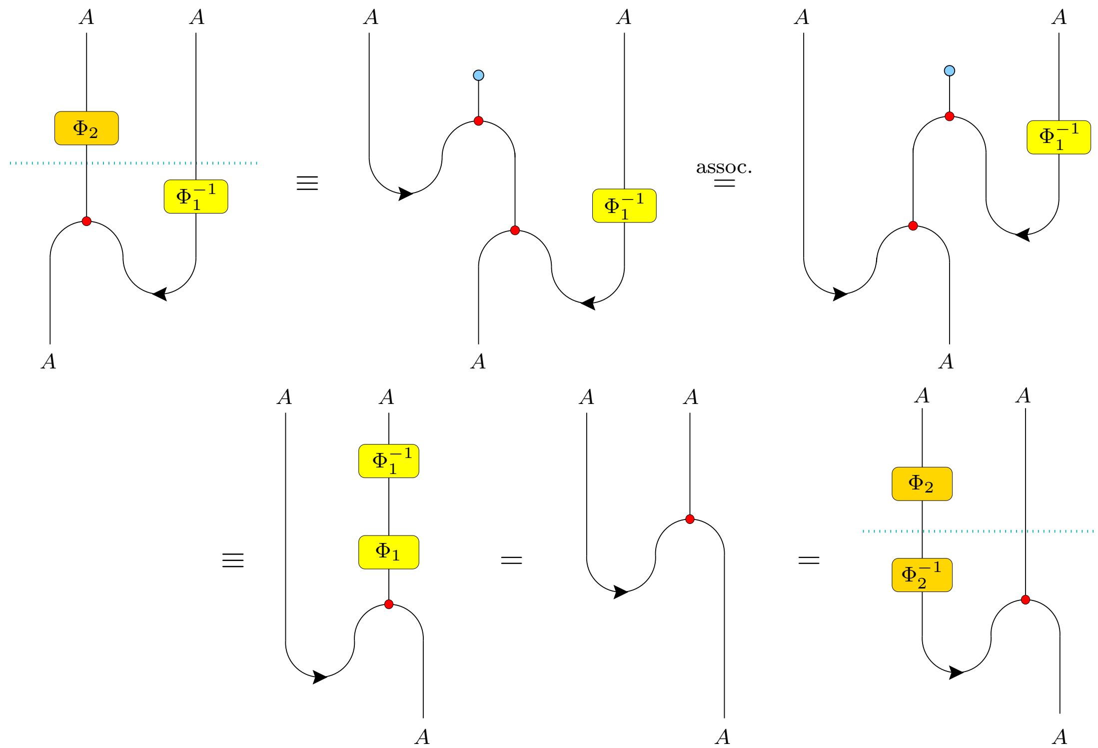
Δ :=


and

Δ' :=


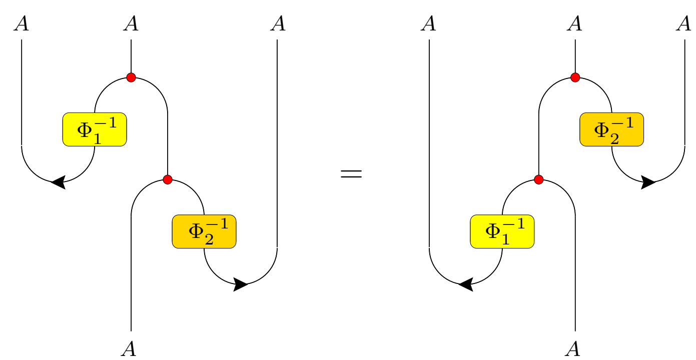
(3.42)

However, the two morphisms Δ and Δ' defined this way actually coincide. To see this, we compose the morphisms as given by (3.40) and (3.41), respectively, with $\Phi_2 \otimes id_A$. In the case of Δ this yields immediately the morphism $(id_{A^\vee} \otimes m) \circ (\tilde{b}_A \otimes id_A)$, while in the case of Δ' the same result follows with the help of associativity:



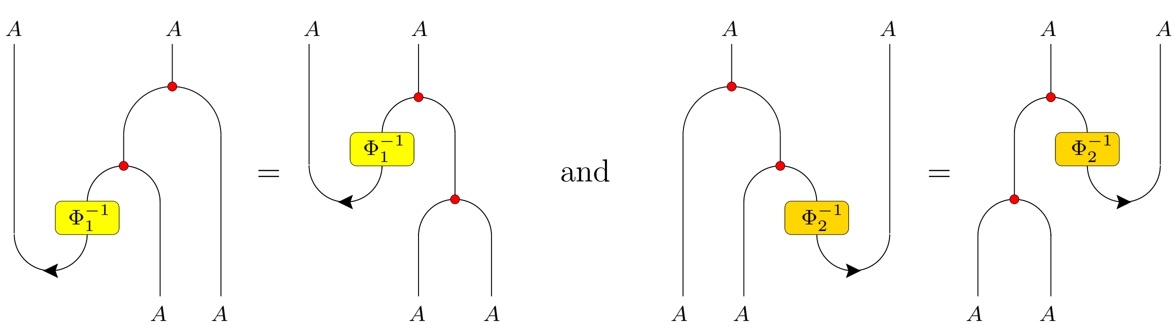
(3.43)

The coassociativity property then follows from the fact that associativity of m implies equality of $(\Delta \otimes id_A) \circ \Delta'$ and $(id_A \otimes \Delta') \circ \Delta$:



(3.44)

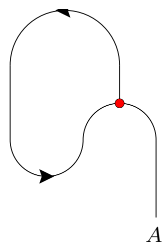
That the counit properties $(id_A \otimes \varepsilon) \circ \Delta = id_A$ and $(\varepsilon \otimes id_A) \circ \Delta' = id_A$ are satisfied follows directly from the definition of Φ_1 and Φ_2. Finally, the Frobenius property (3.29) follows again easily from associativity of m, using Δ for showing the first equality, and Δ' for the second:

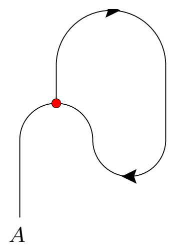


(3.45)

✓

Definition 3.8:
For A an algebra in a sovereign tensor category, the morphisms $\varepsilon_\natural$ and ε_ρ in $\mathrm{Hom}(A,\mathbf{1})$ are defined as

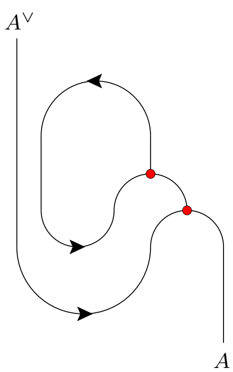
$\varepsilon_\natural :=$ 

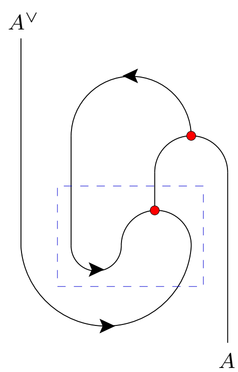
and $\varepsilon_\rho :=$ 

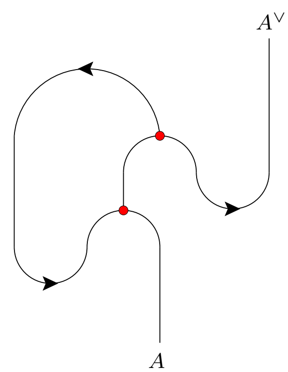
(3.46)

Lemma 3.9:
Let A be an algebra in a sovereign tensor category. Then we have:
(i) A together with $\varepsilon := \xi\,\varepsilon_\natural$ is a symmetric algebra, for any $\xi \in \mathbb{C}$. The same holds with $\varepsilon = \xi\varepsilon_\rho$.
(ii) If A is a symmetric Frobenius algebra, then $\varepsilon_\natural = \varepsilon_\rho$.

Proof:
(i) According to definition 3.4(ii) we must show that $\Phi_2 = \Phi_1$. This is verified as follows. We have

$\Phi_2 \;=\; \xi$ 

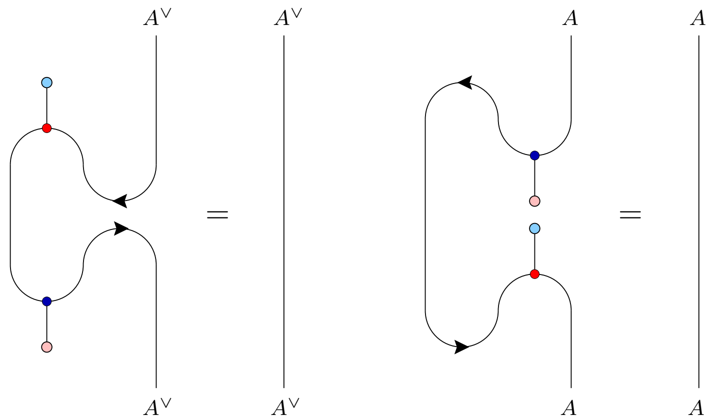
$=\; \xi$ 

$=\; \xi$ 

(3.47)

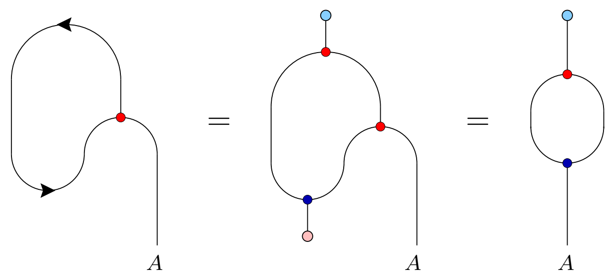
Here the first step is the definition, the second uses associativity, and the third identity follows by replacing the left-dual of the morphism that is enclosed in a dashed box by its right-dual, which is allowed because we are working in a sovereign category. That the right hand side of (3.47) equals Φ_1 follows by using once more associativity and the definition of $\varepsilon = \xi\varepsilon_\natural$.
(ii) Since A is a Frobenius algebra, the morphisms $\Phi_{1,2}$ are invertible. And since A is symmetric, we have $\Phi_1 = \Phi_2$. Thus in particular the equalities $\Phi_1 \circ \Phi_2^{-1} = \Phi_2 \circ \Phi_1^{-1} = id_{A^\vee}$ and

$\Phi_1^{-1}\circ\Phi_2 = \Phi_2^{-1}\circ\Phi_1 = id_A$ hold. Pictorially, we have



(3.48)

as well as the mirror images of these identities. Consider now the transformations



(3.49)

where in the first step we used (3.48) to insert unit and counit, while the second step is an application of the Frobenius and unit properties. Analogously one shows a mirror version of (3.49); combining the two identities then establishes the lemma. ✓

Corollary 3.10:
If A is a haploid algebra, then it is symmetric for any choice of $\varepsilon \in \mathrm{Hom}(A,\mathbf{1})$.

Proof:
By definition, for haploid algebras the morphism space $\mathrm{Hom}(A,\mathbf{1})$ has dimension one. Furthermore, $\varepsilon_\natural \neq 0$ since $\varepsilon_\natural \circ \eta = \dim(A)$. Thus $\varepsilon_\natural$ forms a basis of $\mathrm{Hom}(A,\mathbf{1})$ and any choice of ε is proportional to $\varepsilon_\natural$. ✓

As an immediate consequence of the relation (3.48), for every symmetric special Frobenius algebra the normalisations $\beta_{\mathbf{1}}$ and β_A in (3.31) obey

$$\beta_{\mathbf{1}}\,\beta_A = \dim(A)\,. \tag{3.50}$$

In the present section we keep these normalisations explicitly. But from section 4 onwards we will simplify the presentation by assuming (without loss of generality) that the coproduct is normalised such that $\beta_A = 1$, and hence $\beta_{\mathbf{1}} = \dim(A)$.

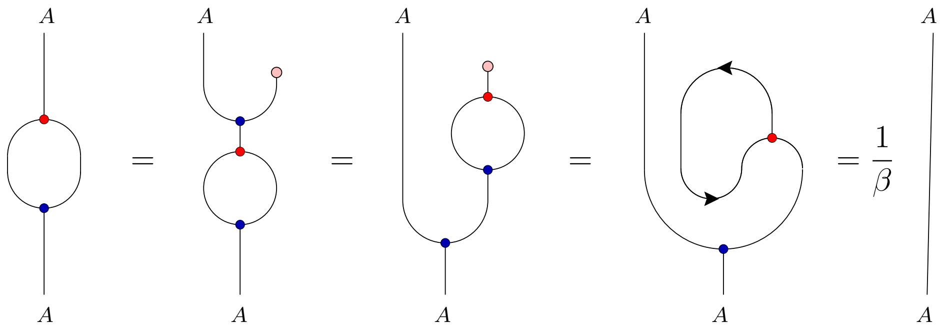
Lemma 3.11:
For a symmetric Frobenius algebra A the following statements are equivalent:
(i) A is special.

(ii) The counit obeys [12] $\varepsilon = \beta\, \varepsilon_\natural$ for some non-zero number β.

Proof:
(i) $\Rightarrow$ (ii) : Composing the specialness property $m \circ \Delta = \beta_A\, id_A$ (where β_A is non-zero) with ε yields $\varepsilon = \beta_A^{-1} \varepsilon \circ m \circ \Delta$, which is the last expression in (3.49). Backtracking the steps in (3.49) thus gives $\varepsilon = \beta_A^{-1} \varepsilon_\natural$.
(ii) $\Rightarrow$ (i) : By composition with the unit we get

$$\varepsilon \circ \eta = \beta\, \varepsilon_\natural \circ \eta = \beta \dim(A) \neq 0\,, \tag{3.51}$$

where in the last step the unit property is used to obtain the trace of id_A. Second, consider the moves



(3.52)

In the first step a counit is inserted, the second step combines the Frobenius property and coassociativity, the third step is again backtracking the steps (3.49), and finally the assumption $\varepsilon = \beta \varepsilon_\natural$ is inserted and the counit property is used. ✓

Lemma 3.12 :
For any algebra (A, m, η) the following two statements are equivalent:
(i) There exist $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ and $\Delta \in \mathrm{Hom}(A, A \otimes A)$ such that $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric special Frobenius algebra.
(ii) The morphism $\Phi_1' \in \mathrm{Hom}(A, A^\vee)$, defined as in (3.32), but with $\varepsilon_\natural$ (as defined in (3.46)) in place of ε, is invertible.
(The same holds when taking ε_ρ instead of $\varepsilon_\natural$).

Proof:
(i) $\Rightarrow$ (ii) : When A is symmetric special Frobenius, we have $\varepsilon = \beta \varepsilon_\natural$ (by lemma 3.11), and hence $\Phi_1 = \beta \Phi_1'$, with a non-zero number β. Further, since A is Frobenius, the morphism Φ_1 is invertible, and hence so is Φ_1'.
(ii) $\Rightarrow$ (i) : We must find morphisms ε and Δ such that $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric special Frobenius algebra. For ε we choose $\varepsilon := \varepsilon_\natural$. From lemma 3.9(i) it follows that A is symmetric. Since by assumption Φ_1 is invertible, there exists a Δ that turns A into a Frobenius algebra; this morphism Δ is the one given in (3.42). Thus $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric Frobenius algebra. But because of $\varepsilon = \varepsilon_\natural$, lemma 3.11 implies that A is special, too. ✓

Remark 3.13 :
Let us add two comments on the case where $\mathcal{C}$ is the category of complex vector spaces.

[12] Recall from lemma 3.9 that $\varepsilon_\natural = \varepsilon_\rho$ for symmetric Frobenius algebras.

(i) Suppose A is an algebra over $\mathbb{C}$ with basis $\{b_i\}$. Then the morphism $\varepsilon_\natural$ is the trace of the multiplication operator, in the sense that, with the product given by $b_i \times b_j = \sum_k m_{ij}^{\;k}\, b_k$, it satisfies $\varepsilon_\natural(b_i) = \sum_k m_{ki}^{\;k}$. For example, when A is the algebra of functions over some finite set, then a basis is given by the delta functions, for which the product reads $b_i \times b_j = \delta_{i,j} b_j$, and we obtain $\varepsilon_\natural(b_i) = 1$. Thus in this case $\varepsilon_\natural$ is an integral with a measure that weighs every point evenly.

(ii) It is easy to construct algebras (A, m, η) such that Φ_1 is not invertible when setting $\varepsilon = \varepsilon_\natural$. Take e.g. $A = \mathbb{C}^2$ with a basis $\{e, n\}$ such that e acts as the unit element while $n \times n = 0$. Then $\varepsilon_\natural(e) = 2$ and $\varepsilon_\natural(n) = 0$, and as a consequence $\Phi_1(n) = 0$.

Proof of theorem 3.6:

(i) Given an algebra object (A, m, η), suppose there exist ε and Δ such that $(A, m, \eta, \Delta, \varepsilon)$ is a symmetric special Frobenius algebra. By lemma 3.11 we have $\varepsilon = \xi \varepsilon_\natural$ for some $\xi \in \mathbb{C}^\times$. But $\varepsilon_\natural$ is already fixed in terms of the multiplication m and thus ε is fixed up to a normalisation constant ξ in terms of (A, m, η). By lemma 3.7 the coproduct Δ is uniquely fixed in terms of $(A, m, \eta, \varepsilon)$ and by lemma 3.9, $(A, m, \eta, \varepsilon)$ is symmetric.

(ii) In section 3.2 it was shown that (A, m, η) with A given in terms of the CFT data by (3.13) and multiplication m defined in terms of boundary structure constants by (3.14) is an algebra object. The counit ε was defined in terms of boundary one-point functions on the upper half plane and has the property (3.23). Note that the calculation leading to (3.23) is based on the assumption that there are no states of negative conformal weight; this assumption is fulfilled in every unitary CFT. The invertibility of the matrix $G(a)$ in (3.16) is equivalent to the statement that the morphism Φ_1 in (3.33) is invertible. Furthermore, comparing (3.23) to ε_ρ, as defined in (3.46) and expressed in a basis, shows that $\varepsilon = \mathrm{const}'\, \varepsilon_\rho$. The constant is non-zero since Φ_1 is invertible. Thus lemma 3.12 is applicable, since $\Phi_1' = (\mathrm{const}')^{-1} \Phi_1$ is invertible as well. ✓

3.4 The associated topological algebra

An algebra object A in a tensor category $\mathcal{C}$ defines automatically also an algebra over $\mathbb{C}$, to be denoted as A_{top} and called the *topological algebra associated to A*. The complex algebra A_{top} is defined as follows. As a vector space, A_{top} is the morphism space

$$A_{\mathrm{top}} := \mathrm{Hom}(\mathbf{1}, A)\,. \tag{3.53}$$

The multiplication m_{top} and unit η_{top} (regarded as a map $\mathbb{C} \to A_{\mathrm{top}}$) are given by

$$m_{\mathrm{top}}(\alpha \otimes \beta) := m \circ (\alpha \otimes \beta)\,, \qquad \eta_{\mathrm{top}}(1) := \eta\,, \tag{3.54}$$

where $\alpha, \beta \in A_{\mathrm{top}}$.

In the CFT interpretation introduced in section 3.2, A_{top} is the algebra of boundary fields (on a fixed boundary condition) that transform in the vacuum representation of the chiral algebra. The fields of weight zero in this set form a topological subsector of the boundary theory, hence the name A_{top}. The presence of such a subsector fits well with the description [88,89,90,91,92,77] of two-dimensional lattice TFTs via symmetric special Frobenius algebras in the category of complex vector spaces.

In the sequel we will deduce various somewhat technical results, which will be instrumental later on. The reader not interested in the proofs of these results may proceed directly to section 3.5.

Let us choose bases for $A_{\rm top}$ and its dual in such a way that

$$b_i \in \mathrm{Hom}(\mathbf{1},A)\,, \qquad b^j \in \mathrm{Hom}(A,\mathbf{1})\,, \qquad b^j \circ b_i = \delta_i^{\,j}\,. \tag{3.55}$$

Lemma 3.14:
Let A be a symmetric special Frobenius algebra in $\mathcal{C}$ and $A_{\rm top}$ its associated topological algebra. Setting

$$\Delta_{\rm top}(\alpha) := \sum_{i,j} \big[(b^i{\otimes}b^j)\circ\Delta\circ\alpha\big]\; b_i{\otimes}b_j\,, \qquad \text{and} \qquad \varepsilon_{\rm top}(\alpha) := \varepsilon\circ\alpha \tag{3.56}$$

for all $\alpha\,{\in}\,A_{\rm top}$ turns $A_{\rm top}$ into a symmetric Frobenius algebra over $\mathbb{C}$.

Proof:
The statement follows by direct computation. The calculations all boil down to the observation that a graph with $A_{\rm top}$-lines, products and coproducts *without loops* and with arbitrary basis elements b_i and b^j inserted at the external lines is equal to the same graph with A-lines, products and coproducts. ✓

Note, however, that with $\varepsilon_{\rm top}$ defined as above, $A_{\rm top}$ is not necessarily special. Indeed, the definition of specialness contains graphs with loops and the above argument no longer applies. For example, let u and v be two non-isomorphic simple objects in $\mathcal{C}$ and let $A\,{=}\,(u{\oplus}v){\otimes}(u{\oplus}v)^\vee$, with product as defined in formula (3.3), and $\varepsilon\,{=}\,\varepsilon_\natural$ as counit. A basis of $A_{\rm top}$ is given by the duality morphisms $\{\tilde b_u,\tilde b_v\}$ of simple objects u,v. One verifies that (with x and y standing for any of u,v) $m_{\rm top}(\tilde b_x,\tilde b_y)\,{=}\,\delta_{x,y}\tilde b_x$ and $\varepsilon_{\rm top}(\tilde b_x)\,{=}\,\dim(u{\oplus}v)\dim(x)$. However, by the same reasoning as in remark 3.13 one shows that $\varepsilon_{\natural,\rm top}(b_x)\,{\equiv}\,1$. It follows that for $\dim(u)\,{\ne}\,\dim(v)$, $\varepsilon_{\rm top}$ as defined by (3.56) is not proportional to $\varepsilon_{\natural,\rm top}$, and hence by lemma 3.11 $A_{\rm top}$ cannot be special.

Definition 3.15:
Let A be an algebra in $\mathcal{C}$ and $A_{\rm top}$ the associated topological algebra. The *relative center of* $A_{\rm top}$ *with respect to* A is the subspace

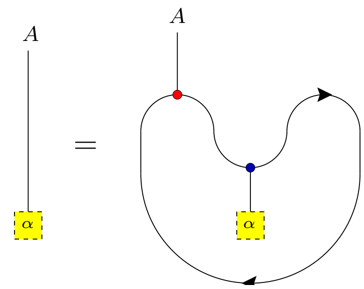
$$\mathrm{cent}_A(A_{\rm top}) := \{\, \alpha\,{\in}\,A_{\rm top} \,|\, m\circ(\alpha{\otimes}id_A)\,{=}\,m\circ(id_A{\otimes}\alpha)\,\}\,. \tag{3.57}$$

Elements in $\mathrm{cent}_A(A_{\rm top})$ in particular commute with all elements of $A_{\rm top}$. Thus the relative center is a subalgebra of the ordinary center $\mathrm{cent}(A_{\rm top})$ of $A_{\rm top}$ (defined in the usual way for algebras over $\mathbb{C}$). Also note that the unit η always lies in $\mathrm{cent}_A(A_{\rm top})$. Thus we have the inclusions

$$\{\xi\eta\,|\,\xi\,{\in}\,\mathbb{C}\} \subseteq \mathrm{cent}_A(A_{\rm top}) \subseteq \mathrm{cent}(A_{\rm top}) \subseteq A_{\rm top}\,. \tag{3.58}$$

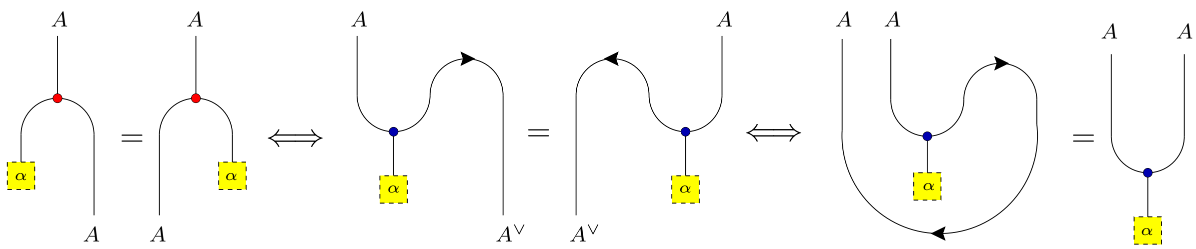
The subset cent_A turns out to be useful in the description of the torus partition function. Indeed, we will see that $Z_{00}\,{=}\,\dim(\mathrm{cent}_A(A_{\rm top}))$. The proof will be given in section 5.3; it relies on the following lemma.

Lemma 3.16:
For any symmetric special Frobenius algebra A with $\beta_A = 1$ one has

$\alpha \in \mathrm{cent}_A(A_{\mathrm{top}}) \quad \Longleftrightarrow$  (3.59)

Proof:

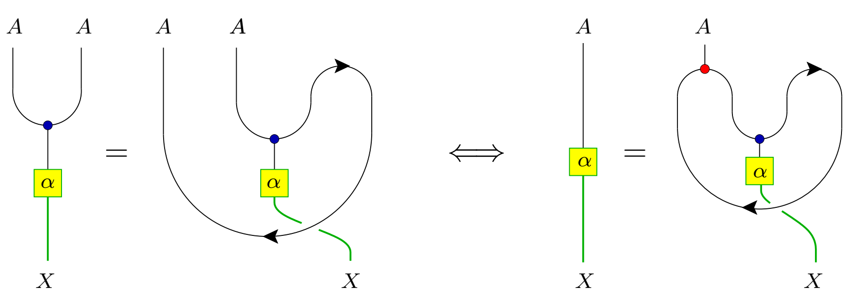
We have the following equivalences:

 (3.60)

The first of these equivalences is obtained by composing the first equality on the left with Φ_2^{-1} on the left and with Φ_1^{-1} on the right hand side (these inverses exist by the Frobenius property, and they are equal by the symmetry property) and then using the Frobenius property, while the second equivalence follows upon composition of the middle identity with a duality morphism. The lemma is now established if the equality on the right hand side of (3.59) can be shown to be equivalent to the one on the right hand side of (3.60). But this is just a special case, obtained by setting $X = \mathbf{1}$, of lemma 3.17 below. ✓

In the following considerations the braiding between the algebra A and arbitrary objects X plays a role.

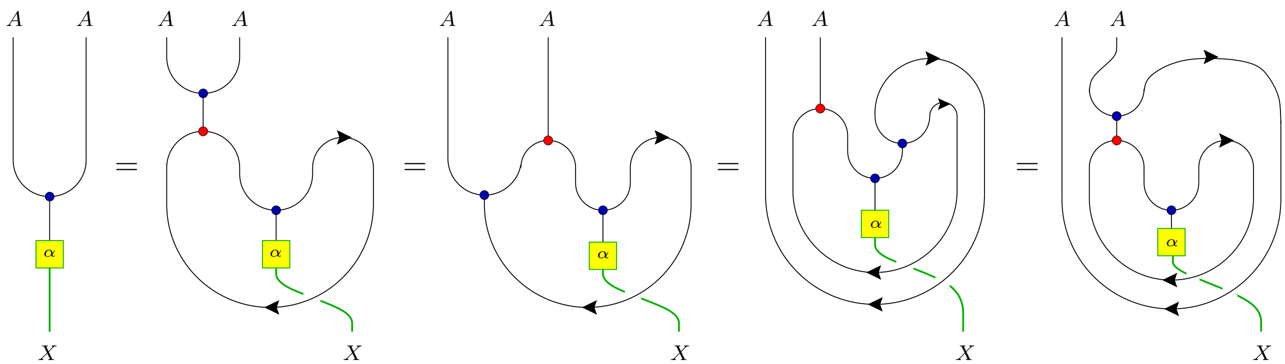
Lemma 3.17:
Let A be a special Frobenius algebra with $\beta_A = 1$. For any object X and any morphism $\alpha \in \mathrm{Hom}(X, A)$ we have

 (3.61)

as well as the analogous equivalence in which all braidings are replaced by inverse braidings.

Proof:
With the braidings chosen as in picture (3.61), the two directions are shown as follows:

⇒ : This is seen immediately by composing both sides of the equality on the left hand side of (3.61) with m and then using specialness of A.
⇐ : Starting from the right hand side of (3.61), and using the Frobenius and (co-)associativity properties, we have



(3.62)

If we now substitute again the equality on the right hand side of (3.61) in the last expression, we arrive at the equality on the left hand side of (3.61). For the inverse braidings the equivalence is shown in the same way. ✓

3.5 Sums, products, and the opposite algebra

Given two (symmetric special Frobenius) algebras A and B in a ribbon category, we can define algebra structures on $A \oplus B$ and $A \otimes B$. It is also possible to twist the product of A by the braiding, giving rise to an algebra A_{op}. In section 5.3 we will encounter a physical interpretation of these operations: Denoting the torus partition function obtained from an algebra A as $Z(A)$, and defining $\tilde{Z}(A)_{kl} := Z(A)_{\bar{k}l}$, we have the matrix equations $\tilde{Z}(A \oplus B) = \tilde{Z}(A) + \tilde{Z}(B)$, $\tilde{Z}(A \otimes B) = \tilde{Z}(A)\, \tilde{Z}(B)$ and $Z(A_{\text{op}}) = Z(A)^{\text{t}}$.

These expressions suggest the interpretation that $A \oplus B$ describes a superposition of two CFTs – in the sense introduced before formula (3.25) – while $A \otimes B$ defines a (in general non-commutative) product of two CFTs that are associated to the same chiral data. Note that this product is different from the usual product $\text{CFT}_1 \times \text{CFT}_2$, where the new stress tensor is given by $T_1 + T_2$ and thus the central charges add. The CFT resulting from $A \otimes B$ has the same stress tensor and central charge as the ones resulting from A and B.

In the rest of this section we will make precise the notions of $A \oplus B$, $A \otimes B$ and A_{op} and prove some properties of these algebras.

Proposition 3.18 :
[*Opposite algebra*]
(i) $A = (A, m, \eta)$ is an algebra if and only if $A_{\text{op}} := (A, m \circ (c_{A,A})^{-1}, \eta)$ is an algebra. [13]

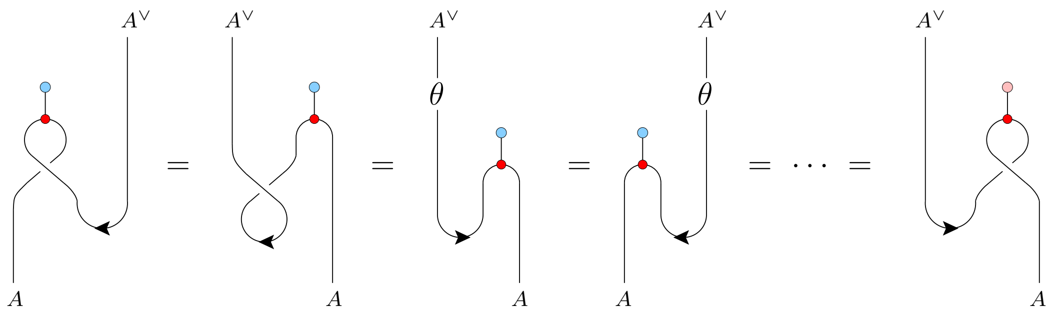
[13] Note that we take the inverse braiding in this definition. It is this choice that is needed to prove propositions 4.6 and 5.3. The first of these links the definition of A_{op} to that of the tensor product, while the second links that of the tensor product to that of the graph for the torus partition function. In this sense the conventions implicit in the graph (5.30) fix the convention for A_{op}.

(ii) $A = (A, m, \eta, \Delta, \varepsilon)$ is a symmetric special Frobenius algebra if and only if

$$A_{\rm op} := (A, m \circ (c_{A,A})^{-1}, \eta, c_{A,A} \circ \Delta, \varepsilon) \tag{3.63}$$

is a symmetric special Frobenius algebra.

Proof:
The statement (i) follows by a straightforward application of definition 3.1. Similarly, for obtaining (ii) one checks easily from the definitions in 3.2, 3.3 and 3.4 that the respective properties of A and $A_{\rm op}$ follow from each other. For example, that A symmetric implies $A_{\rm op}$ symmetric is seen as follows:



(3.64)

The first morphism is Φ_1 for $A_{\rm op}$, expressed via the multiplication m of A. Symmetry of A enters in the third equality. ✓

Note that even though $A_{\rm op}$ is equal to A as an object in $\mathcal{C}$, it has a different multiplication and is therefore, in general, not isomorphic to A. In particular, this remark still applies when A and $A_{\rm op}$ are symmetric – recall that the symmetry property, introduced in definition 3.4(ii), does not refer in any way to the braiding.

Corollary 3.19:
(i) For any algebra (A, m, η) and any $n \in \mathbb{Z}$ also $A^{(n)} := (A, m \circ (c_{A,A})^n, \eta)$ is an algebra.
(ii) For any symmetric special Frobenius algebra $(A, m, \eta, \Delta, \varepsilon)$ and any $n \in \mathbb{Z}$ also

$$A^{(n)} := (A, m \circ (c_{A,A})^n, \eta, (c_{A,A})^{-n} \circ \Delta, \varepsilon) \tag{3.65}$$

is a symmetric special Frobenius algebra.

In particular we have $A = A^{(0)}$ and $A_{\rm op} = A^{(-1)}$.

Proposition 3.20:
(i) The twist θ_A is an intertwiner between the algebras $A^{(n)}$ and $A^{(n+2)}$, in the sense that

$$\theta_A \circ \eta^{(n)} = \eta^{(n+2)} \qquad \text{and} \qquad \theta_A \circ m^{(n)} = m^{(n+2)} \circ (\theta_A \otimes \theta_A)\,, \tag{3.66}$$

with $m^{(n)} = m \circ (c_{A,A})^n$ and $\eta^{(n)} = \eta$ the product and unit of $A^{(n)}$.
(ii) If $A^{(n)}$ is symmetric special Frobenius then in addition

$$\varepsilon^{(n)} = \varepsilon^{(n+2)} \circ \theta_A \qquad \text{and} \qquad (\theta_A \otimes \theta_A) \circ \Delta^{(n)} = \Delta^{(n+2)} \circ \theta_A\,. \tag{3.67}$$

Proof:
By direct computation. One only has to use some of the defining properties of a modular tensor category, namely (see (2.10–2.11)) that $\theta_V \circ f = f \circ \theta_U$, for any $f \in \mathrm{Hom}(U,V)$ as well as $\theta_{U\otimes V} = c_{V,U} \circ c_{U,V} \circ (\theta_U \otimes \theta_V)$. For instance,

$$\theta_A \circ m^{(n)} = m^{(n)} \circ \theta_{A\otimes A} = m^{(n)} \circ (c_{A,A})^2 \circ (\theta_A \otimes \theta_A) = m^{(n+2)} \circ (\theta_A \otimes \theta_A)\,. \tag{3.68}$$

✓

By definition of the direct sum $A \oplus B$ of two objects A and B, there exist morphisms

$$X_C \in \mathrm{Hom}(C, A\oplus B) \qquad \text{and} \qquad Y_C \in \mathrm{Hom}(A\oplus B, C) \tag{3.69}$$

for $C \in \{A,B\}$ such that

$$Y_C \circ X_D = \delta_{C,D}\, id_C \qquad \text{and} \qquad X_A \circ Y_A + X_B \circ Y_B = id_{A\oplus B}\,. \tag{3.70}$$

for $C, D \in \{A,B\}$. When A and B are algebras, then these morphisms can be used to endow $A \oplus B$ with the structure of an algebra, too:

Proposition 3.21:
[*Direct sum of algebras*]
(i) The triple $(A\oplus B, m_{A\oplus B}, \eta_{A\oplus B})$ with

$$m_{A\oplus B} := \sum_{C=A,B} X_C \circ m_C \circ (Y_C \otimes Y_C) \qquad \text{and} \qquad \eta_{A\oplus B} := \sum_{C=A,B} X_C \circ \eta_C \tag{3.71}$$

furnishes an algebra structure on the object $A\oplus B$.
(ii) When A, B are symmetric special Frobenius and normalised such that $\varepsilon_A \circ \eta_A = \dim(A)$ and $\varepsilon_B \circ \eta_B = \dim(B)$, then $A\oplus B$ is symmetric special Frobenius, with counit and coproduct given by

$$\varepsilon_{A\oplus B} = \sum_{C=A,B} \varepsilon_C \circ Y_C \qquad \text{and} \qquad \Delta_{A\oplus B} = \sum_{C=A,B} (X_C \otimes X_C) \circ \Delta_C \circ Y_C\,. \tag{3.72}$$

Proof:
The proof proceeds again by direct computation. To check, for example, that $A\oplus B$ is special, one computes

$$\begin{aligned} m_{A\oplus B} \circ \Delta_{A\oplus B} &= \Big[\sum_{C=A,B} X_C \circ m_C \circ (Y_C \otimes Y_C)\Big] \circ \Big[\sum_{D=A,B} (X_D \otimes X_D) \circ \Delta_D \circ Y_D\Big] \\ &= \sum_C X_C \circ m_C \circ \Delta_C \circ Y_C = \sum_C X_C \circ id_C \circ Y_C = id_{A\oplus B}\,. \end{aligned} \tag{3.73}$$

In the second equality we insert the orthogonality relation in (3.70), while the third step uses that A and B are special, with constants β_A and β_B both equal to 1. The last step is just the completeness in (3.70). ✓

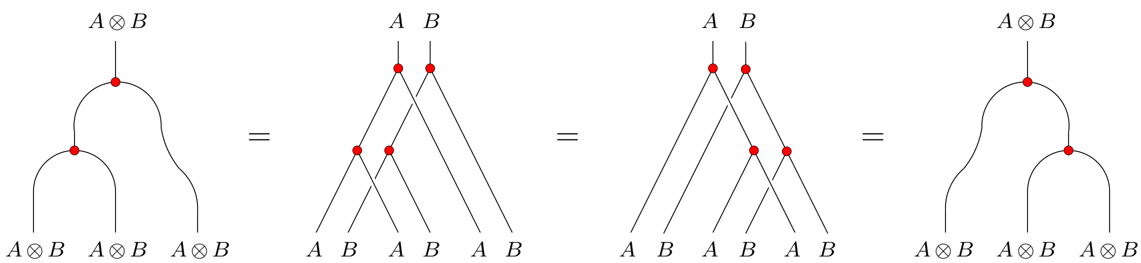
Proposition 3.22:
[*Tensor product of algebras*]
(i) For any two algebras A and B the triple $(A\otimes B, m_{A\otimes B}, \eta_{A\otimes B})$ with

$$m_{A\otimes B} := (m_A \otimes m_B) \circ (id_A \otimes (c_{A,B})^{-1} \otimes id_B) \qquad \text{and} \qquad \eta_{A\otimes B} := \eta_A \otimes \eta_B \tag{3.74}$$

furnishes an algebra structure on the object $A\otimes B$.
(ii) If A, B are symmetric special Frobenius, then also $A\otimes B$ is symmetric special Frobenius, with counit and coproduct given by

$$\varepsilon_{A\otimes B} = \varepsilon_A \otimes \varepsilon_B \qquad \text{and} \qquad \Delta_{A\otimes B} = (id_A \otimes c_{A,B} \otimes id_B) \circ (\Delta_A \otimes \Delta_B)\,. \tag{3.75}$$

Proof:
In this case, too, one checks all the properties by direct computation. To give an example, the associativity of $A\otimes B$ follows from



(3.76)

✓

Remark 3.23:
(i) In the definition (3.74) one could have used the braiding $c_{B,A}$ instead of $(c_{A,B})^{-1}$. It is easy to check that this alternative multiplication on the object $A\otimes B$ coincides with the multiplication on $(A_{\rm op}\otimes B_{\rm op})^{(1)}$ given above. We will not introduce a separate symbol for this way of defining a multiplication on $A\otimes B$.
(ii) For any three algebras A, B, C there are two a priori distinct ways to construct an algebra structure on the object $A\otimes B\otimes C$ by the tensor product of algebras, namely $(A\otimes B)\otimes C$ and $A\otimes(B\otimes C)$. As $\mathcal{C}$ is a strict tensor category, the two combinations are identical as objects in $\mathcal{C}$. One can quickly convince oneself from the definitions in proposition 3.22 that the unit and multiplication morphism (as well as the counit and comultiplication, if they exist) are identical in both cases as well. Thus the tensor product of algebras defined in proposition 3.22 is associative.
(iii) As will be discussed in a forthcoming paper, the operations of taking sums, products and opposites of algebras are compatible with Morita equivalence. Denote by $[A]$ the Morita class of A, and let A' be another representative of $[A]$, and similarly for $[B]$ and B, B'. Then

$$[A\oplus B] = [A'\oplus B']\,, \qquad [A\otimes B] = [A'\otimes B'] \qquad \text{and} \qquad [A_{\rm op}] = [A'_{\rm op}]\,. \tag{3.77}$$

As a consequence, the operations on algebras can be lifted to the level of CFT. We will return to this issue in section 5.3.

It is also worth mentioning that the resulting ring structure on Morita classes of symmetric special Frobenius algebras is reminiscent of Brauer groups for (finite-dimensional, central, semisimple) algebras over a field, compare e.g. chapter 4 of [24]. Indeed, consider the tensor category Vect(k) of vector spaces over some field k, which we do not assume to be algebraically closed. Then the module categories over Vect(k) – or, equivalently, the Morita classes of associative algebras in Vect(k) – are classified [13] by division algebras over k, which amounts to compute the Brauer group of all finite extensions of k. Quite generally, the problem of classifying full CFTs based on a modular tensor category $\mathcal{C}$ can therefore be expected to amount to the following two tasks: Find all extensions of $\mathcal{C}$, and compute the (suitably defined) Brauer group of each extended theory. Here we call a modular tensor category $\mathcal{D}$ an *extension* of $\mathcal{C}$ iff there exists a haploid commutative symmetric special Frobenius algebra A in $\mathcal{C}$ such that $\mathcal{D}$ is equivalent to the modular tensor category of local A-modules (see section 4.1 below).

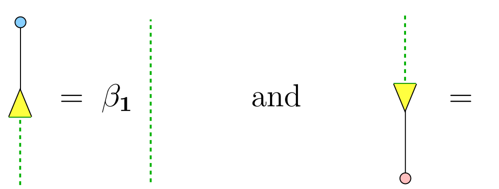
3.6 The case ${N_{ij}}^k \in \{0,1\}$ and $\dim \mathrm{Hom}(U_k, A) \in \{0,1\}$

To continue our meta example, we will further limit ourselves to the case where each simple subobject a in the algebra A occurs with multiplicity one. In this case we can omit the index labelling the basis of possible embeddings of a in (3.4), rendering the notation less heavy.

An algebra object is now described by a collection of pairwise non-isomorphic simple objects $\{\mathbf{1}, U_{a_1}, \dots, U_{a_n}\}$. The multiplication on A can be expressed by constants ${m_{ab}}^c$, as in formula (3.7), and a comultiplication similarly by constants Δ_c^{ab}. The associativity condition (3.8) takes the form

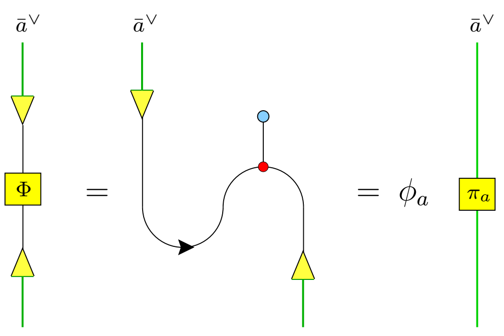
$$ {m_{ab}}^f\, {m_{fc}}^d = \sum_{e \prec A} {m_{bc}}^e\, {m_{ae}}^d\, \mathsf{F}^{(a\,b\,c)\,d}_{e\,f} \,, \tag{3.78} $$

Here we use the symbol "$\prec$" to indicate the relation "is a simple subobject of". Further, we choose the basis vector in Hom($\mathbf{1}, A$) to be given by the unit η of A; then the dual basis vector in Hom($A, \mathbf{1}$) is fixed to be the multiple $\varepsilon/\beta_{\mathbf{1}}$ of the counit ε. Thus in particular composing with the counit and unit, respectively, gives

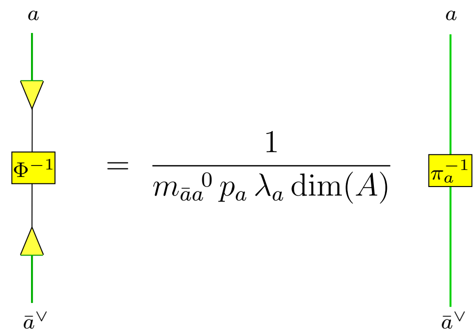


(3.79)

Now the multiplication already determines the comultiplication. Let us make this relation explicit. The normalisation (3.79) of the unit implies that ${m_{a0}}^a = 1 = {m_{0a}}^a$. Since A is haploid, the two morphisms Φ_1 and Φ_2 in (3.33) are automatically equal. For them to be isomorphisms we need ${m_{a\bar a}}^0 \neq 0$ for all $a \prec A$. To determine the inverse of $\Phi \equiv \Phi_1 = \Phi_2$, we express it in a basis as

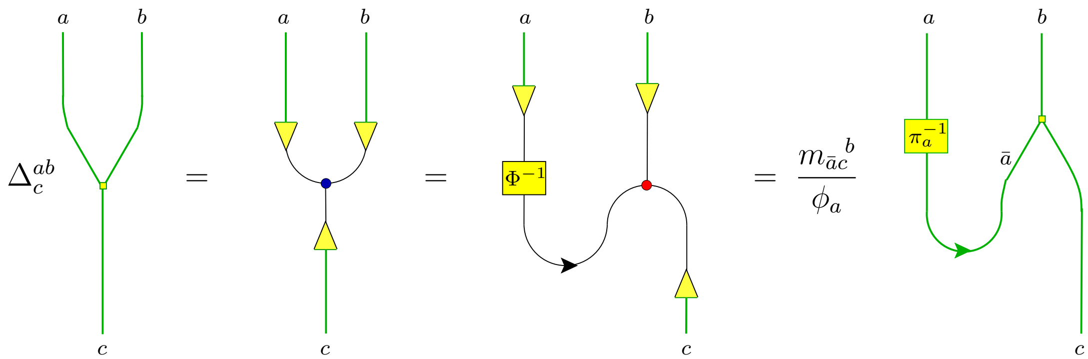


with $\quad \phi_a = {m_{\bar a a}}^0\, p_a\, \lambda_a \dim(A)\,, \quad$ (3.80) where p_a and λ_a are the numbers introduced in (2.24) and (2.34), respectively. It follows that the map Φ^{-1} is then just given by



(3.81)

Using the expression (3.40) for the coproduct and the identities (2.60), one finds



(3.82)

With the help of the relations (2.34), (2.35), (2.46) and (2.60), it then follows that

$$\Delta_c^{ab} = \frac{1}{\dim(A)} \cdot \frac{m_{\bar a c}{}^{b}}{m_{\bar a a}{}^{0}} \cdot \frac{\mathsf{F}^{(a\,\bar a\,c)\,c}_{\;b\,0}}{\mathsf{F}^{(a\,\bar a\,a)\,a}_{\;0\,0}}\,. \tag{3.83}$$

3.6.1 Example: Free boson

Different algebra objects in the $\mathbb{Z}_{2N}$ free boson modular category correspond to different compactification radii. (This will be made more explicit in section 5.6.1 below, when we come to computing the torus partition function.) There is one algebra object associated to every subgroup of $\mathbb{Z}_{2N}$ that contains only 'even elements', in the sense that there is one algebra object for every divisor r of N. For such a subgroup, $2r$ with $0 \le r < N$ is its 'minimal' element. We denote as A_{2r} the corresponding object

$$A_{2r} := \bigoplus_{n=0}^{N/r-1} [2nr]\,. \tag{3.84}$$

The multiplication on A_{2r} is given by

$$m_{[a][b]}{}^{[a+b]} = 1 \qquad \text{for all} \quad a,b,c \prec A\,. \tag{3.85}$$

One checks that all non-zero F-matrix elements (2.77) with only even labels are equal to 1. As a consequence, with (3.85) the associativity condition (3.78) is satisfied trivially, as $1\,{=}\,1$.

The algebras presented above do not exhaust *all* algebra objects of the free boson theory, but they do give all *haploid* ones. This follows from a general treatment of algebra objects containing only simple currents as simple subobjects, which will be presented in a separate paper. For now we only remark that simple current theory [93, 94, 27, 95] tells us that to obtain a modular invariant, a simple current J must be [96] in the so-called *effective center*, i.e. the product of its order (the smallest natural number ℓ such that $J^{\otimes \ell} \cong \mathbf{1}$) and its conformal weight must be an integer. In the case under study the simple current orbits are generated by objects r with r a divisor of $2N$, i.e. by simple currents of order $2N/r$. Modulo integers, the conformal weight of r is $r^2/4N$ and thus it is in the effective center iff $r \in 2\mathbb{Z}$.

3.6.2 Example: E_7 modular invariant

According to [32, 13], for the E_7 type modular invariant of the $\mathfrak{su}(2)$ WZW model at level 16, the associated algebra object A should be expected to have the form

$$A = (0) \oplus (8) \oplus (16) . \tag{3.86}$$

This object can indeed be turned into an algebra [32, 13], and we will show in the sequel how this can be established with our methods. But of course, at this point we do not yet know whether the modular invariant of the CFT associated to the algebra (3.86) is of E_7 type – this will be verified in section 5.6.2.

Note that $J := (16)$ is a simple current in $\mathfrak{su}(2)_{16}$ and that $f := (8)$ is its fixed point. Let us therefore investigate, more generally, algebra objects of the form

$$A = \mathbf{1} \oplus f \oplus J , \tag{3.87}$$

for which the fusion rules of the simple subobjects are

$$J \otimes J \cong \mathbf{1} , \qquad J \otimes f \cong f \qquad \text{and} \qquad f \otimes f \cong \mathbf{1} \oplus f \oplus J \oplus \cdots . \tag{3.88}$$

Whereas for the free boson one would have easily guessed the product structure on A, in the E_7 case at this point we do have to find a solution to *polynomial* equations. However, we will see (though not in all detail in the present paper) that all the rest of the calculations, down to the structure constants, then reduces to solving systems of *linear* equations. Let us write the relevant equations explicitly as relations that the numbers ${m_{ab}}^c$ must satisfy in order to be the components of a valid multiplication morphism. First, according to lemma 3.6(ii) the Frobenius property imposes the restriction ${m_{aa}}^{\mathbf{1}} \neq 0$; we can therefore choose the basis elements in $\mathrm{Hom}(f, A)$ and $\mathrm{Hom}(J, A)$ such that

$${m_{ff}}^{\mathbf{1}} = {m_{JJ}}^{\mathbf{1}} = 1 . \tag{3.89}$$

The system (3.78) of polynomial equations encoding associativity is finite, and it is not difficult to write out all equations when A is of the particular form (3.87). This system of conditions is necessary and sufficient for the numbers ${m_{ab}}^c$ to provide an algebra structure. After some manipulations, one deduces that these conditions imply

$$\begin{aligned} &{m_{ff}}^J = {m_{fJ}}^f = {m_{Jf}}^f , \qquad ({m_{ff}}^J)^2 = \mathsf{F}^{(J\,f\,f)\,J}_{\mathbf{1}\,f} , \\ &({m_{ff}}^f)^2 = \big[\mathsf{F}^{(f\,f\,f)\,f}_{f\,\mathbf{1}}\big]^{-1} \big[1 - \mathsf{F}^{(f\,f\,f)\,f}_{\mathbf{1}\,\mathbf{1}} - \mathsf{F}^{(J\,f\,f)\,J}_{\mathbf{1}\,f} \mathsf{F}^{(f\,f\,f)\,f}_{J\,\mathbf{1}}\big] . \end{aligned} \tag{3.90}$$

A priori this fixes the algebra structure up to sign choices only. But in fact the convention (3.89) still permits a sign flip both in Hom(f,A) and in Hom(J,A). As a result, the choice of sign allowed by (3.90) for the numbers ${m_{ff}}^f$ and ${m_{ff}}^J$ can be absorbed into a change of basis, and hence (3.90) determines the algebra structure uniquely. Thus, if it exists, the algebra structure on A is unique up to isomorphism.

Note that in (3.90) it is assumed that $\mathsf{F}_{f\,\mathbf{1}}^{(f\,f\,f)\,f} \neq 0$ (which requires in particular that $f \prec f \star f$). This holds true for $\mathfrak{su}(2)_{16}$, but it need not hold in general. (If not, then the calculation will proceed differently from what is reported in the sequel; we do not discuss this case here.) In the same spirit we will also assume that ${m_{ff}}^f \neq 0$. Again this may not be satisfied in some examples, but it does hold for $\mathfrak{su}(2)_{16}$. With these two assumptions, the remaining associativity constraints are equivalent to

$$\begin{aligned}
&\nu_J = \nu_f = 1\,, \qquad \mathsf{F}_{f\,f}^{(J\,f\,J)\,f} = \mathsf{F}_{f\,f}^{(f\,J\,f)\,J} = 1\,,\\
&\mathsf{F}_{f\,f}^{(J\,f\,f)\,f} = \mathsf{F}_{f\,f}^{(f\,J\,f)\,f} = \mathsf{F}_{f\,f}^{(f\,f\,J)\,f} = \mathsf{F}_{f\,f}^{(f\,f\,f)\,J} = 1\,,\\
&\mathsf{F}_{\mathbf{1}\,f}^{(J\,f\,f)\,J}\mathsf{F}_{f\,\mathbf{1}}^{(J\,J\,f)\,f} = 1\,, \qquad \mathsf{F}_{\mathbf{1}\,f}^{(J\,f\,f)\,J} = \mathsf{F}_{\mathbf{1}\,f}^{(f\,J\,J)\,f}\,, \qquad \mathsf{F}_{f\,\mathbf{1}}^{(J\,J\,f)\,f} = \mathsf{F}_{f\,\mathbf{1}}^{(f\,f\,J)\,J}\,,\\
&\mathsf{F}_{\mathbf{1}\,k}^{(f\,f\,f)\,f} + ({m_{ff}}^J)^2\,\mathsf{F}_{J\,k}^{(f\,f\,f)\,f} + ({m_{ff}}^f)^2\,\mathsf{F}_{f\,k}^{(f\,f\,f)\,f} = ({m_{ff}}^k)^2 \quad \text{for } k \in \{J,f\}\,,\\
&\mathsf{F}_{\mathbf{1}\,k}^{(f\,f\,f)\,f} + ({m_{ff}}^J)^2\,\mathsf{F}_{J\,k}^{(f\,f\,f)\,f} + ({m_{ff}}^f)^2\,\mathsf{F}_{f\,k}^{(f\,f\,f)\,f} = 0 \quad \text{for } k \in \mathcal{I}\setminus\{\mathbf{1},J,f\} \text{ and } N_{ff}^{\ \ k} = 1\,.
\end{aligned} \tag{3.91}$$

Owing to the pentagon identity fulfilled by the fusing matrices, not all of these requirements are independent. At first sight it might look surprising that the pentagon identity for morphisms from $f \otimes J \otimes J \otimes f$ to f implies that J has Frobenius–Schur indicator $\nu_J = 1$ (recall formula (2.47)). However, on general grounds [97], for self-dual f and J the fact that ${N_{Jf}}^f$ is 1 already implies that $\nu_J \nu_f = \nu_f$.

But not all of the relations (3.91) follow from the pentagon, as one can easily convince oneself by finding explicit counter examples. So $A = \mathbf{1} \oplus f \oplus J$ can only be endowed with an algebra structure in rather special cases, as befits the fact that it should describe an exceptional modular invariant. We verified numerically that the relations (3.91) hold true for $\mathfrak{su}(2)_k$ with $k = 16$. In fact they also hold for $k = 8$ for which they yield the D-invariant (see section 5.6.2 below), but not for various other k we tested. For $k = 4$ the object $\mathbf{1} \oplus f \oplus J$ possesses an algebra structure, too (it yields just the A-invariant), but with ${m_{ff}}^f = 0$ so that the associativity constraints look different from (3.91).

4 Representation theory and boundary conditions

4.1 Representations and modules

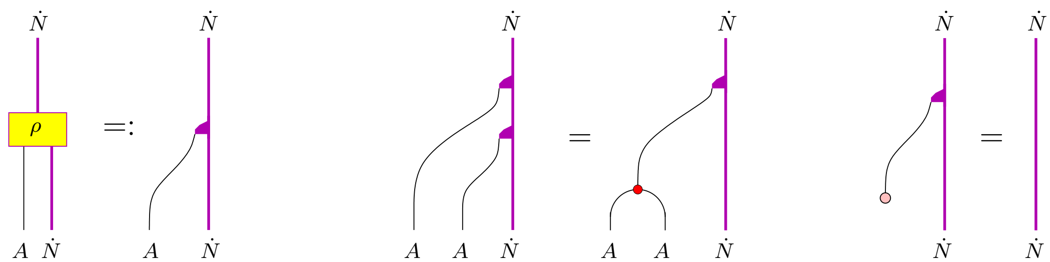
When studying an ordinary algebra over the complex numbers (or any other number field), a key concept is that of a representation of the algebra, together with the closely related notion of a module, i.e. the vector space on which a representation acts. Moreover, the modules of an algebra (or analogously, of a group, a Lie algebra, a quantum group or similar algebraic structures) are the objects of a category, with morphisms given by algebra (or group, Lie algebra, ...) intertwiners.

It is not difficult to extend these notions in such a way that they apply to algebra objects in arbitrary tensor categories $\mathcal{C}$. The basic notion is that of an A-module:

Definition 4.1:
For A an algebra in a tensor category $\mathcal{C}$, a (left) A-*module* in $\mathcal{C}$ is a pair $N = (\dot{N}, \rho)$ of two data: an object $\dot{N}$ of $\mathcal{C}$ and a morphism of $\mathcal{C}$ that specifies the action of A on $\dot{N}$ – the *representation morphism* $\rho \equiv \rho_N \in \mathrm{Hom}(A \otimes \dot{N}, \dot{N})$. Further, ρ must satisfy the *representation properties*

$$\rho \circ (m \otimes id_{\dot{N}}) = \rho \circ (id_A \otimes \rho) \qquad \text{and} \qquad \rho \circ (\eta \otimes id_{\dot{N}}) = id_{\dot{N}} \,. \tag{4.1}$$

Pictorially:



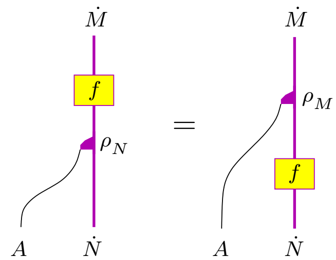
(4.2)

What is introduced here is called a *left* module because the action of A on N is from the left. Analogously one can define right A-modules, on which A acts from the right, i.e. for which the representation morphism is an element of $\mathrm{Hom}(\dot{N} \otimes A, \dot{N})$. And there is also the notion of a (left or right) *comodule* over a co-algebra, involving a morphism in $\mathrm{Hom}(\dot{N}, A \otimes \dot{N})$ (or $\mathrm{Hom}(\dot{N}, \dot{N} \otimes A)$, respectively) that satisfies relations corresponding to the pictures (4.2) turned upside-down, with coproduct and counit in place of product and unit. Note that not every object $U \in \mathcal{O}bj(\mathcal{C})$ needs to underlie some A-module, and that an object of $\mathcal{C}$ can be an A-module in several inequivalent ways.

Among the morphisms f between two objects $\dot{N}, \dot{M} \in \mathcal{O}bj(\mathcal{C})$ that both carry the structure of an A-module, those that intertwine the action of A play a special role. Here the notion of intertwining the A-action is analogous as for modules over ordinary algebras:

Definition 4.2:
For (left) A-modules N, M, an A-*intertwiner* is a morphism f between $\dot{N}$ and $\dot{M}$ satisfying

$f \circ \rho_N = \rho_M \circ (id_A \otimes f)$, i.e.



(4.3)

Given an algebra A in $\mathcal{C}$, taking the A-modules in $\mathcal{C}$ as objects and the subspaces

$$\mathrm{Hom}_A(N, M) := \{ f \in \mathrm{Hom}(\dot{N}, \dot{M}) \,|\, f \circ \rho_N = \rho_M \circ (id_A \otimes f) \} \tag{4.4}$$

of the $\mathcal{C}$-morphisms that intertwine the A-action as morphisms results in another category, called the *category of* (left) *A-modules* and denoted by $\mathcal{C}_A$. Similarly as for morphisms in $\mathcal{C}$ (see formula (2.3)), we will use the shorthand notation

$$\dim \mathrm{Hom}_A(M, N) =: \langle M, N \rangle_A \tag{4.5}$$

for any two A-modules $M, N \in \mathcal{O}bj(\mathcal{C}_A)$.

Typical representation theoretic tools, like induced modules and reciprocity theorems, generalise to the category theoretic setting (see e.g. [87, 98, 36, 13]) and allow one to work out the representation theory in concrete examples. In particular, one shows that $\mathcal{C}_A$ inherits various properties of $\mathcal{C}$. For instance, when $\mathcal{C}$ is modular and hence semisimple and when A is special Frobenius, then the category $\mathcal{C}_A$ is semisimple. On the other hand, that $\mathcal{C}$ is modular does *not* imply that $\mathcal{C}_A$ is modular. In fact, it does not even imply that $\mathcal{C}_A$ is a tensor category.

However, a sufficient condition for $\mathcal{C}_A$ to be tensor is then that A is a commutative algebra and has trivial twist, i.e. $\theta_A = id_A$. The modules over such an algebra A in a modular tensor category fall into two different classes, the *local* and the *solitonic* modules. Local modules M can be characterised [87] by the fact that their twist is a morphism in $\mathrm{Hom}_A(M, M)$. If M is simple it follows that θ_M is a multiple of the identity, i.e. all simple subobjects of M have the same conformal weight modulo integers. Let us denote by $(\mathcal{C}_A)_{\mathrm{loc}}$ the full subcategory of $\mathcal{C}_A$ whose objects are the local A-modules. One can check that the twist and braiding on $\mathcal{C}$ induce a twist and braiding on $(\mathcal{C}_A)_{\mathrm{loc}}$. Moreover, if $\mathcal{C}$ is a modular category, then $(\mathcal{C}_A)_{\mathrm{loc}}$ is modular as well [87].[14] In CFT terms, $(\mathcal{C}_A)_{\mathrm{loc}}$ is the modular category for the chiral algebra $\mathfrak{V}$ extended by the (primary) chiral fields that correspond to the simple subobjects of the algebra A. Local modules correspond to boundary conditions that preserve the extended chiral algebra, whereas solitonic modules break the extended symmetry (while still preserving $\mathfrak{V}$) [101, 102, 103].

A simple object in the category $\mathcal{C}_A$ is called a *simple A-module*. Similarly as in section 2.1 (see before formula (2.22)), let us introduce a set of definite representatives for the isomorphism classes of simple A-modules; we denote them by M_κ and the corresponding label set by

$$\mathcal{J} = \{\kappa\}\,. \tag{4.6}$$

As we will see later on, in the case of interest to us there are only finitely many isomorphism classes of simple A-modules, i.e. $\mathcal{J}$ is a finite set.

[14] The corresponding subcategory of $\mathcal{C}$ is not modular, but only 'pre-modular'. It can be shown [99, 100] that such a category possesses a unique 'modularisation'; this modularisation is precisely $(\mathcal{C}_A)_{\mathrm{loc}}$.

That the category $\mathcal{C}_A$ of A-modules is in general not a tensor category even when the category $\mathcal{C}$ is modular, actually does not come as a surprise. The crucial property of $\mathcal{C}_A$ is that it is a so-called module category over $\mathcal{C}$, and tensoriality is not a natural ingredient of that structure. A *module category* [79, 52, 104, 13] $\mathcal{M}$ over a tensor category $\mathcal{C}$ is a category together with an exact bifunctor [15]

$$\otimes: \quad \mathcal{M} \otimes \mathcal{C} \to \mathcal{M} \tag{4.7}$$

that obeys generalised unit and associativity constraints: There are natural isomorphisms from $(M \otimes V) \otimes W$ to $M \otimes (V \otimes W)$ and from $M \otimes \mathbf{1}$ to M, where M is an object of $\mathcal{M}$ while V and W are objects of $\mathcal{C}$.

In the case at hand, where $\mathcal{M} = \mathcal{C}_A$ is the category of left A-modules, the structure of a module category amounts to the observation that for any left A-module M and any object V of $\mathcal{C}$, also $M \otimes V$ has a natural structure of left A-module. Now recall that the tensor product bifunctor of $\mathcal{C}$ is exact, implying that the Grothendieck group $K_0(\mathcal{C})$ inherits from the tensor product on $\mathcal{C}$ the structure of a ring, the fusion ring. It follows from the exactness of the bifunctor (4.7) that the Grothendieck group $K_0(\mathcal{M})$ carries the structure of a right module over the ring $K_0(\mathcal{C})$. Thus the notion of a module category over a tensor category is a categorification of the algebraic notion of a module over a ring, in the same sense as the structure of a tensor category categorifies the structure of a ring.

It can be shown [13] that every module category over a tensor category $\mathcal{C}$ is equivalent to the category of left modules for some associative algebra A in $\mathcal{C}$. Moreover, the analysis (see section 3.2, and section 4.4 below) of the OPE of boundary fields, including those that change boundary conditions, shows that any unitary rational conformal field theory that possesses at least one boundary condition preserving the chiral algebra gives rise to a module category over the category of Moore–Seiberg data. This provides another way to understand the emergence of algebra objects which are central to our approach. (Indeed, the reconstruction theorem of [13] allows one to recover not just one algebra in $\mathcal{C}$, but rather a family of Morita equivalent algebras.)

We are now also in a position to explain why the algebras of open string states corresponding to different boundary conditions are Morita equivalent. The object M of the module category $\mathcal{M}$ that describes a boundary condition is not necessarily simple, i.e. we can allow for a superposition of elementary boundary conditions. According to theorem 1 in [13], each such object M gives rise to an algebra A_M in the underlying tensor category $\mathcal{C}$, which is constructed using the so-called internal Hom:

$$A_M := \underline{\mathrm{Hom}}(M, M)\,. \tag{4.8}$$

In the same theorem it is shown that the category of A_M-modules is equivalent to the module category $\mathcal{M}$. Thus in particular, any two objects M_1 and M_2 of the module category, i.e. any two boundary conditions, give rise to algebras A_{M_1} and A_{M_2} with equivalent categories of left modules. By definition 10 of [13], these two algebras of open string states are therefore Morita equivalent. In fact, the two interpolating bimodules can be given explicitly in terms of internal Hom's as well, namely as $\underline{\mathrm{Hom}}(M_1, M_2)$ and $\underline{\mathrm{Hom}}(M_2, M_1)$.

In any module category $\mathcal{M}$, one can consider the morphism spaces $\mathrm{Hom}_{\mathcal{M}}(M \otimes V, N)$ for any pair M, N of objects of $\mathcal{M}$ and any object V of $\mathcal{C}$. When A is a special Frobenius algebra, as in the case of our interest, then semisimplicity of $\mathcal{C}$ implies that $\mathcal{M} = \mathcal{C}_A$ is semisimple as well. One can then generalise the arguments given in section 3.2 and choose definite bases of

[15] It is conventional to denote this functor by the same symbol as the tensor product bifunctor of $\mathcal{C}$.

the spaces $\mathrm{Hom}_{\mathcal{M}}(M\otimes V,N)$ with M,N simple modules and V a simple object, and then write down the matrix elements of the isomorphism

$$\begin{aligned}&\bigoplus_{p\in\mathcal{I}}\mathrm{Hom}_{\mathcal{M}}(M_i\otimes U_p,M_l)\otimes\mathrm{Hom}_{\mathcal{C}}(U_j\otimes U_k,U_p)\\ &\qquad\cong\ \mathrm{Hom}_{\mathcal{M}}(M_i\otimes U_j\otimes U_k,M_l)\ \cong\ \bigoplus_{q\in\mathcal{J}}(M_q\otimes U_k,M_l)\otimes\mathrm{Hom}(M_i\otimes U_j,M_q)\end{aligned}\tag{4.9}$$

in these bases. Using the same graphs as in (2.36), but replacing the simple objects U_i, U_l and U_q of $\mathcal{C}$ by simple objects M_i, M_l and M_q in the module category, one arrives at a generalisation of the 6j-symbols (F-matrices). These symbols, often denoted by ${}^{(1)}F$, are labelled by three simple objects of $\mathcal{M}$ and three simple objects of $\mathcal{C}$ (as well as one basis morphism of the type (2.29) and three other basis morphisms involving also modules). These mixed 6j-symbols express the associativity of the bifunctor (4.7). They appear naturally in the theory of weak Hopf algebras [8]. As already mentioned after formula (3.12), in CFT they arise as structure constants for the operator products of boundary fields.

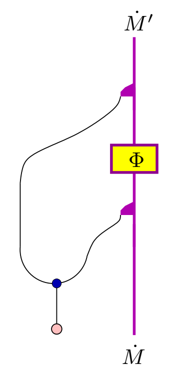
Below it will be convenient to use the following construction of A-intertwiners:

Definition 4.3:
For any $\Phi\in\mathrm{Hom}(\dot M,\dot M')$, with M,M' (left) A-modules, the A-*averaged morphism* $\overline{\Phi}$ is

$$\overline{\Phi}:=\rho_{M'}\circ[id_A\otimes(\Phi\circ\rho_M)]\circ[(\Delta\circ\eta)\otimes id_{\dot M}]\,.\tag{4.10}$$

Pictorially:

$$\overline{\Phi}\ =\tag{4.11}$$



Lemma 4.4:
(i) When A is a Frobenius algebra, then for every $\Phi\in\mathrm{Hom}(\dot M,\dot M')$, where M,M' are A-modules, the A-averaged morphism $\overline{\Phi}$ is an A-intertwiner,

$$\overline{\Phi}\in\mathrm{Hom}_A(M,M')\,.\tag{4.12}$$

(ii) When A is special Frobenius and $\Phi\in\mathrm{Hom}_A(\dot M,\dot M')$, then $\overline{\Phi}=\Phi$.
(iii) When A is a symmetric special Frobenius algebra, then the trace of an endomorphism does not change under averaging,

$$\mathrm{tr}\,\overline{\Phi}=\mathrm{tr}\,\Phi\,.\tag{4.13}$$

Proof:
(i) The statement follows by a simple application of (both parts of) the Frobenius property of

A.
(ii) Start by using the fact that Φ is an A-intertwiner, so as to move the representation morphism ρ_M past Φ to get another $\rho_{M'}$. Next employ the representation property for $\rho_{M'}$. The A-lines can then be removed completely by first using specialness of A (recall the convention $\beta_A = 1$) and the the unit property for $\rho_{M'}$.
(iii) Start with the left trace, as defined in (2.14) (recall that since $\mathcal{C}$ is modular and hence spherical, $\mathrm{tr}_\mathrm{r} = \mathrm{tr}_\mathrm{l} =: \mathrm{tr}$). Owing to the cyclicity property of the trace we can pull the upper A-line around until the two representation morphisms are directly composed. Now use symmetry of A as in (3.35) and the first representation property (4.1). Finally employ specialness (3.31) of A and the second representation property in (4.1). Upon these manipulations the A-lines have completely disappeared. ✓

We will also need the notion of an A-B-bimodule. This is the following structure:

Definition 4.5:
For A and B algebra objects in a tensor category $\mathcal{C}$, an A-B-*bimodule* M is a triple $(\dot M, \rho^A, \tilde\rho^B)$, consisting of an object $\dot M \in \mathcal{O}bj(\mathcal{C})$ and two morphisms $\rho^A \in \mathrm{Hom}(A \otimes \dot M, \dot M)$ and $\tilde\rho^B \in \mathrm{Hom}(\dot M \otimes B, \dot M)$, such that $(\dot M, \rho^A)$ is a left A-module, $(\dot M, \tilde\rho^B)$ is a right B-module, and such that the actions ρ^A and $\tilde\rho^B$ commute, i.e.

$$\rho^A \circ (id_A \otimes \tilde\rho^B) = \tilde\rho^B \circ (\rho^A \otimes id_B)\,. \tag{4.14}$$

When we want to make explicit the algebras over which M is a bimodule, we write it as ${}_AM_B$. If $\mathcal{C}$ is a braided tensor category, then bimodules can equivalently be thought of as left $A{\otimes}B_\mathrm{op}$-modules, see remark 12 in [13].

This is an important point, because it implies that we do not need to develop new techniques to find all bimodules, once we know how to deal with left modules. In particular the methods of induced modules described in section 4.3 below can be applied.

Let us formulate the correspondence between bimodules and left modules more precisely.

Proposition 4.6:
For A, B algebras in a braided tensor category, the mapping (see picture (4.19) below)

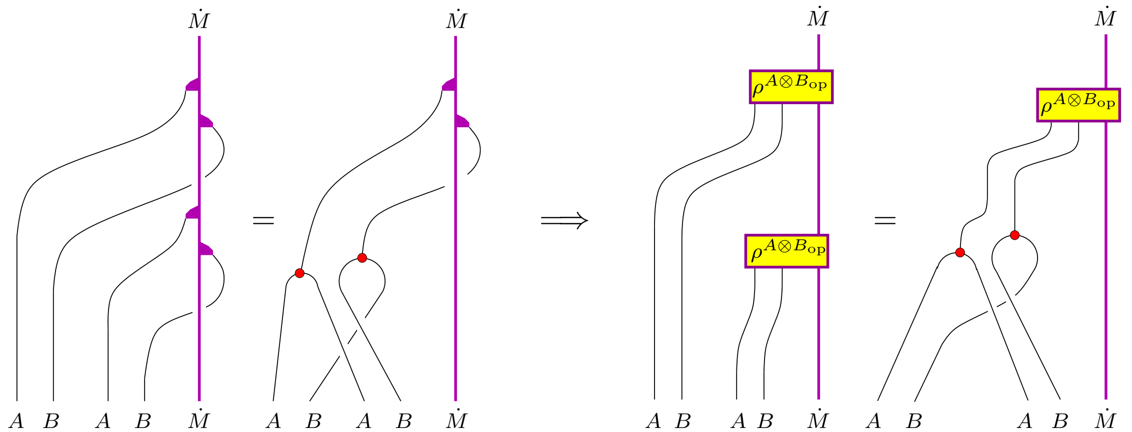
$$f: \quad (\dot M, \rho^A, \tilde\rho^B) \;\mapsto\; (\dot M, \rho^A \circ (id_A \otimes \tilde\rho^B) \circ (id_A \otimes c^{-1}_{\dot M,B})) \tag{4.15}$$

takes A-B-bimodules to left $A{\otimes}B_\mathrm{op}$-modules. f is invertible, with inverse g given by

$$g: \quad (\dot M, \rho^{A\otimes B_\mathrm{op}}) \;\mapsto\; (\dot M, \rho^{A\otimes B_\mathrm{op}} \circ (id_A \otimes \eta_{B_\mathrm{op}} \otimes id_{\dot M}), \rho^{A\otimes B_\mathrm{op}} \circ (\eta_A \otimes c_{\dot M,B}))\,. \tag{4.16}$$

Proof:
To check that $g \circ f = id$ and $f \circ g = id$ is straightforward. Now suppose we are given an A-B-bimodule M. Substituting the prescription (4.15), the representation property for M can be

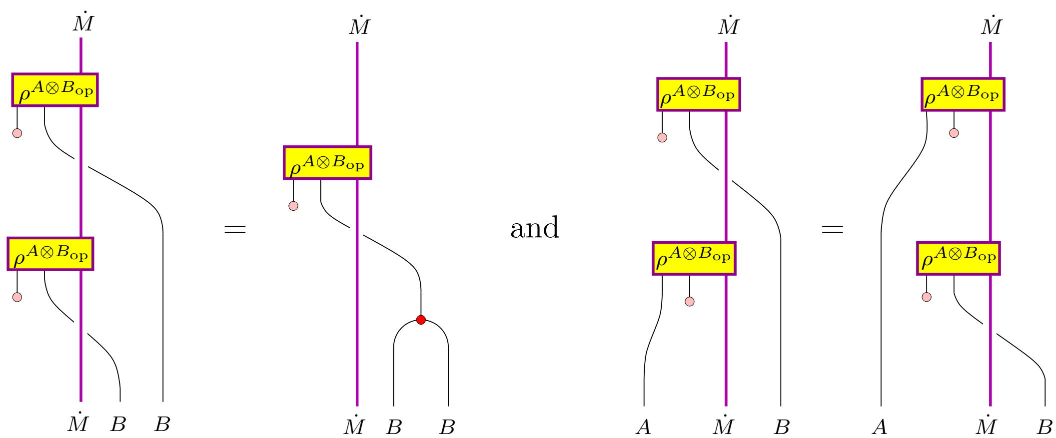
rewritten as in the first equality in



(4.17)

This equality, in turn, directly implies the second equality in (4.17), which precisely spells out the representation property for a left $A{\otimes}B_{\rm op}$-module. (Note that in the second equality, the multiplication morphisms are those of A and B, not of $B_{\rm op}$.)

Conversely, let us suppose that M is a left $A{\otimes}B_{\rm op}$-module, and verify that $g(M)$ provides an A-B-bimodule. It is easily checked that $\rho^{A\otimes B_{\rm op}} \circ (id_A{\otimes}\eta_{B_{\rm op}}{\otimes}id_{\dot M})$ does provide a left representation of A. That $\rho^{A\otimes B_{\rm op}} \circ (\eta_A{\otimes}c_{\dot M,B})$ provides a right representation of B and that the two actions commute amounts to verifying that

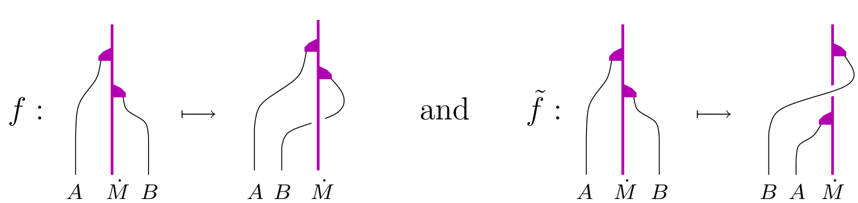


(4.18)

Both of these properties follow from the representation property of left $A{\otimes}B_{\rm op}$-modules as given in the second equality in (4.17). ✓

Remark 4.7:

(i) Proposition 4.6 provides an isomorphism f that takes A-B-bimodules to left $A{\otimes}B^{(-1)}$-modules. In an analogous manner one constructs an isomorphism $\tilde f$ that takes A-B-bimodules to left $B^{(1)}{\otimes}A$-modules. In pictures, the two isomorphisms are given by



(4.19)

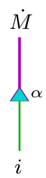
(ii) It can be shown that for any Frobenius algebra A in a tensor category $\mathcal{C}$, the category ${}_A\mathcal{C}_A$ of A-A-bimodules is again a tensor category. This is in remarkable contrast to the category $\mathcal{C}_A$ of left A-modules, which in general cannot can be equipped with a tensor product. Moreover, when $\mathcal{C}$ is sovereign and A is symmetric, then ${}_A\mathcal{C}_A$ is equipped with a duality. However, in general the bimodule category does not have a braiding, and therefore it does not have a twist either.

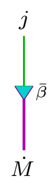
4.2 Representation functions

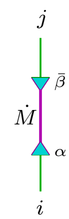
For any given A-module M we choose bases $b^M_{(i,\alpha)}$ for the morphism spaces [16] $\mathrm{Hom}(U_i,\dot M)$ and dual bases $b^{(j,\beta)}_M$ for $\mathrm{Hom}(\dot M,U_j)$ such that

$$b^{(j,\beta)}_M \circ b^M_{(i,\alpha)} = \delta_i^j\, \delta_\alpha^\beta\, \mathrm{id}_{U_i}\,. \tag{4.20}$$

Pictorially:

$b^M_{(i,\alpha)} =:$ 

, $b^{(j,\beta)}_M =:$ 

and 

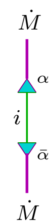
$= \delta_i^j\, \delta_\alpha^\beta$

(4.21)

Dominance in the category $\mathcal{C}$ implies the completeness property

$$\sum_{i\in\mathcal{I}} \sum_\alpha b^M_{(i,\alpha)} \circ b^{(i,\alpha)}_M = id_{\dot M}\,, \tag{4.22}$$

i.e.

$\sum_{i\in\mathcal{I}} \sum_\alpha$ 

$=$ 

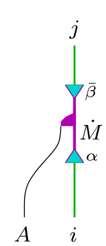
(4.23)

Next let us associate to each pair of morphisms $f\in\mathrm{Hom}(X,\dot M)$ and $g\in\mathrm{Hom}(\dot M,Y)$ the morphism

$$\Psi(f,g) := g\circ\rho_M\circ(id_A\otimes f) \;\in \mathrm{Hom}(A\otimes X,Y)\,. \tag{4.24}$$

We can then introduce the following notion:

Definition 4.8:
(i) For any A-module M the morphism

$\rho_{M(i,\alpha)}^{\ (j,\beta)} := \Psi(b^M_{(i,\alpha)}, b^{(j,\beta)}_M) =$ 

(4.25)

[16] The label $i\in\mathcal{I}$ refers to the chosen representatives U_i of the isomorphism classes of simple objects in $\mathcal{C}$, see section 2.1.

(with $i, j \in \mathcal{I}$ and $b^M_{(i,\alpha)}, b^{(j,\beta)}_M$ basis elements in $\mathrm{Hom}(U_i, \dot M)$ and $\mathrm{Hom}(\dot M, U_j)$ as described above) is called a *representation function* of M.
(ii) For every $j \in \mathcal{I}$, the jth *character* of M is the sum

$$\chi^j_M := \sum_\alpha \rho^{\;(j,\alpha)}_{M(j,\alpha)} \in \mathrm{Hom}(A \otimes U_j, U_j) \tag{4.26}$$

of representation functions, and the jth *dimension morphism* $\mathcal{D}_j$ of M is

$$\mathcal{D}_j(M) := \chi^j_M \circ (\eta \otimes id_{U_j}) \in \mathrm{End}(U_j)\,. \tag{4.27}$$

This diction is sensible because in the special case $\mathcal{C} = \mathrm{Vect}(\mathbb{C})$ and A the group algebra $\mathbb{C}[\mathrm{G}]$ of a group G, the quantities (4.25) are indeed ordinary representation functions $(\rho_M(g))^b_a$. (Also, a module possesses then only a single character, since $\mathrm{Vect}(\mathbb{C})$ has only a single isomorphism class of simple objects.) Besides the representation property

$$\sum_b \rho_M(g)^b_a\, \rho_M(h)^c_b = \rho_M(gh)^c_a\,, \tag{4.28}$$

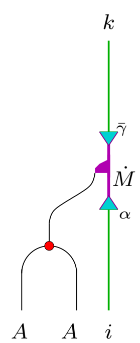
ordinary representation functions for simple modules M, M' also obey the orthogonality relation

$$\frac{1}{|G|} \sum_{g\in G} (\rho_M(g)^*)^b_a\, \rho_{M'}(g)^d_c = \frac{1}{\dim(\dot M)}\, \delta^d_a \delta^b_c\, \delta_{M,M'}\,. \tag{4.29}$$

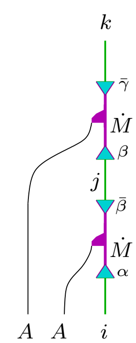
In our context, the representation property amounts to the identity

$$\rho^{\;(k,\gamma)}_{M(i,\alpha)} \circ (m \otimes id_{U_i}) = \sum_{(j,\beta)} \rho^{\;(k,\gamma)}_{M,(j,\beta)} \circ (id_A \otimes \rho^{\;(j,\beta)}_{M,(i\alpha)}) \tag{4.30}$$

in $\mathrm{Hom}(A{\otimes}A{\otimes}U_i, U_k)$, i.e.



$$= \sum_{j\in\mathcal{I}} \sum_\beta \tag{4.31}$$

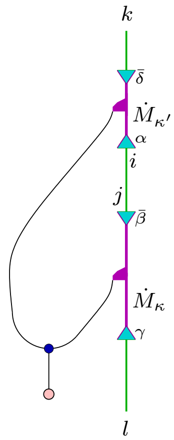


(Use completeness (4.23) and the representation property (4.1) to see this equality.) The generalisation of orthogonality is less direct. We find:

Lemma 4.9:
The representation functions (4.25) for simple modules M_κ, $M_{\kappa'}$ ($\kappa, \kappa' \in \mathcal{J}$) of a special Frobenius algebra A satisfy

$$\rho^{\;(k,\delta)}_{M_{\kappa'}(j,\alpha)} \circ [id_A \otimes \rho^{\;(i,\beta)}_{M_\kappa(l,\gamma)}] \circ [(\Delta \circ \eta) \otimes id_{U_l}] = \frac{\dim(U_i)}{\dim(\dot M_\kappa)}\, \delta_{\kappa,\kappa'}\, \delta^{(i,\alpha)}_{(j,\beta)} \delta^{(k,\delta)}_{(l,\gamma)}\, id_{U_l}\,. \tag{4.32}$$

Pictorially:



$$= \frac{\dim(U_i)}{\dim(\dot M_\kappa)}\, \delta_{\kappa,\kappa'}\, \delta_{(j,\beta)}^{(i,\alpha)}\, \delta_{(l,\gamma)}^{(k,\delta)} \tag{4.33}$$

Proof:
Since U_k and U_l are simple objects (and we have restricted our attention to only a single representative of each isomorphism class), the morphism on the left hand side can be non-zero only for $k=l$. For the same reason we also need $i=j$, and we will tacitly assume this in the sequel. Similarly, noting that the left hand side involves an A-averaged morphism $\overline{\Phi}$ (see definition 4.3) – namely the one associated to $\Phi = b_{(i,\alpha)}^{M_{\kappa'}} \circ b_{M_\kappa}^{(j,\beta)} \in \mathrm{Hom}(\dot M_\kappa, \dot M_{\kappa'})$ – and that by lemma 4.4(i) $\overline{\Phi}$ is an A-intertwiner, we can use the fact that M_κ and $M_{\kappa'}$ are simple modules, and non-isomorphic for $\kappa \neq \kappa'$, to conclude that $M_\kappa = M_{\kappa'} =: M$. Moreover, $\overline{\Phi}$ must then be a multiple $\xi_{\alpha\beta}^i$ of $id_{\dot M}$, and hence the morphism on the left hand side of (4.32) is equal to

$$\varpi = \xi_{\alpha\beta}^i\, b_M^{(l,\delta)} \circ b_{(l,\gamma)}^M\,. \tag{4.34}$$

To determine the coefficients $\xi_{\alpha\beta}^i$ we take the trace of (4.34) and sum over $l \in \mathcal{I}$ and over $\gamma = \delta$ (i.e. over dual bases in $\mathrm{Hom}(U_l, \dot M)$ and $\mathrm{Hom}(\dot M, U_l)$). Using the cyclicity of the trace and then the completeness property (4.23) yields

$$\sum_{l\in\mathcal{I}} \sum_\gamma \mathrm{tr}\,\varpi = \xi_{\alpha\beta}^i \dim(\dot M)\,. \tag{4.35}$$

On the other hand, when applying the trace and summation on the left hand side of (4.32), and then again using the cyclicity and completeness, we get

$$\sum_{l\in\mathcal{I}} \sum_\gamma \mathrm{tr}\,\varpi = \mathrm{tr}\,\overline{\Phi} = \mathrm{tr}\,\Phi = \delta_\alpha^\beta \dim(U_i)\,, \tag{4.36}$$

where in the second step one invokes lemma 4.4(iii) and in the last step once more cyclicity of the trace, as well as the orthogonality (4.21) is used. Comparison of the two results yields $\xi_{\alpha\beta}^i = \delta_\alpha^\beta \dim(U_i)/\dim(\dot M)$ and thereby establishes the validity of (4.32). ✓

Corollary 4.10:
The characters (4.26) of the simple A-modules $M_\kappa, M_{\kappa'}$ satisfy [17]

$$\chi_{M_\kappa}^i \circ (id_A \otimes \chi_{M_{\kappa'}}^i) \circ [(\Delta\circ\eta) \otimes id_{U_i}] = \delta_{\kappa,\kappa'}\, \frac{\dim(U_i)}{\dim(\dot M_\kappa)}\, \langle U_i,\dot M_\kappa\rangle\, id_{U_i}\,, \tag{4.37}$$

[17] Recall the shorthand $\langle X,Y\rangle = \dim \mathrm{Hom}(X,Y)$ introduced in (2.3).

and the dimension morphisms (4.27) obey

$$\mathcal{D}_i(M_\kappa) = \langle U_i,\dot{M}_\kappa\rangle\, id_{U_i} \,. \tag{4.38}$$

Proof:
Formula (4.37) follows immediately from (4.33) by setting $j=k=l=i$, $\alpha=\delta$, $\beta=\gamma$ and summing over α and β. Equation (4.38) is a direct consequence of the unit property of η and the orthogonality (4.21). ✓

Note that in the category of vector spaces, evaluating the character of a module at the unit element yields the dimension of the module. In the situation studied here, $\mathcal{D}_i(M)$ tells us with which multiplicity the simple object U_i "occurs in M", and we obtain the (quantum) dimension of $\dot{M}$ as

$$\dim(\dot{M}) = \sum_{i\in\mathcal{I}} \mathrm{tr}\,\mathcal{D}_i(M) = \sum_{i\in\mathcal{I}} \langle U_i,\dot{M}\rangle \dim(U_i) \,. \tag{4.39}$$

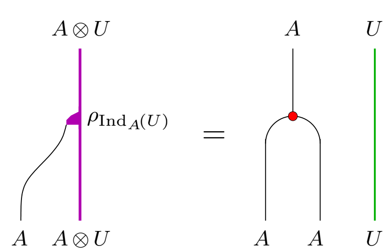
4.3 Induced modules

Later on the A-modules will label the boundary conditions of the CFT that A defines. Given an algebra A in $\mathcal{C}$, we would therefore like to find all A-modules obtainable from objects in $\mathcal{C}$. A powerful method to find A-modules is the one of induced modules, which we recall briefly in this section. The main result is that if A is special Frobenius, then every A-module is a submodule of an induced module. The extension of the method of induced representations to a general category theoretic setting was developed in [87, 36].

Definition 4.11 :
For $U\in\mathcal{O}bj(\mathcal{C})$, the *induced* (left) *module* $\mathrm{Ind}_A(U)$ is the A-module that is equal to $A{\otimes}U$ as an object in $\mathcal{C}$, and has representation morphism $m\otimes id_U$, i.e.

$$\mathrm{Ind}_A(U) := (\, A{\otimes}U \,,\, m\otimes id_U\,)\,. \tag{4.40}$$

Pictorially:



(4.41)

That the morphism (4.40) satisfies the representation properties (4.1) follows directly from the associativity of m and the unit property of η. Analogously, $\widetilde{\mathrm{Ind}}_A(U) = (U{\otimes}A, id_U{\otimes}m)$ is a right A-module, called an induced right module. For the tensor unit $\mathbf{1}$ one obtains this way just the algebra itself, $\mathrm{Ind}_A(\mathbf{1}) = A = \widetilde{\mathrm{Ind}}_A(\mathbf{1})$. Moreover, the product of A is a morphism between $A{\otimes}A$ and A as A-A-bimodules, $m\in\mathrm{Hom}_{A|A}(A{\otimes}A, A)$, while the coproduct Δ is in $\mathrm{Hom}_{A|A}(A, A{\otimes}A)$ if and only if A is Frobenius.

Proposition 4.12:
[*Reciprocity*] For A a Frobenius algebra in a tensor category $\mathcal{C}$, $U \in \mathcal{O}bj(\mathcal{C})$ and M an A-module, there are natural isomorphisms

$$\mathrm{Hom}(U,\dot{M}) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{Ind}_A(U),M) \quad \text{and} \quad \mathrm{Hom}(\dot{M},U) \xrightarrow{\cong} \mathrm{Hom}_A(M,\mathrm{Ind}_A(U))\,, \qquad (4.42)$$

given by $\varphi \mapsto \rho_M \circ (id_A \otimes \varphi)$ for $\varphi \in \mathrm{Hom}(U,\dot{M})$ and $\varphi \mapsto (id_A \otimes (\varphi \circ \rho_M)) \circ ((\Delta \circ \eta) \otimes id_{\dot{M}})$ for $\varphi \in \mathrm{Hom}(\dot{M},U)$. In particular,

$$\langle U,\dot{M}\rangle = \langle \mathrm{Ind}_A(U),M\rangle_A \qquad \text{and} \qquad \langle \dot{M},U\rangle = \langle M,\mathrm{Ind}_A(U)\rangle_A\,. \qquad (4.43)$$

For a proof see e.g. [36], propositions 4.7 and 4.11. [18]

Proposition 4.13:
If A is a special Frobenius algebra in a semisimple tensor category $\mathcal{C}$, then every A-module is a submodule of an induced module, and the category $\mathcal{C}_A$ of A-modules is semisimple.

For a proof see e.g. [36], lemma 5.24 and proposition 5.25.

That $\mathcal{C}_A$ is semisimple means that every A-module can be written as the direct sum of finitely many simple A-modules. Furthermore, every simple module already appears as a submodule of an induced module $\mathrm{Ind}_A(U)$ with U a simple object of $\mathcal{C}$. For $\mathcal{C}$ a modular tensor category, which is the case under study, there are only finitely many isomorphism classes of simple objects, and as a consequence there are then also only finitely many isomorphism classes of simple A-modules. Thus as announced, the index set $\mathcal{J}$ (4.6) is finite.

One important aspect of proposition 4.12 is that it allows us to learn already quite a lot about the structure of simple A-modules from the fusion ring of $\mathcal{C}$ alone. In simple cases – like for the free boson and for the algebra giving the E_7 modular invariant, which are treated in section 4.5 below – the reciprocity relation already determines how many simple A-modules there are and into which simple objects of $\mathcal{C}$ they decompose as elements of $\mathcal{O}bj(\mathcal{C})$.

Even when U is a simple object of $\mathcal{C}$, the induced module $\mathrm{Ind}_A(U)$ will in general *not* be a simple A-module. Indeed, we obtain:

Corollary 4.14:
(i) The module $\mathrm{Ind}_A(U_j)$ $(j \in \mathcal{I})$ decomposes into simple modules M_κ $(\kappa \in \mathcal{J})$ according to

$$\mathrm{Ind}_A(U_j) \;\cong\; \bigoplus_{\kappa\in\mathcal{J}} \langle U_j,\dot{M}_\kappa\rangle\, M_\kappa\,. \qquad (4.44)$$

(ii) For every simple object U of $\mathcal{C}$ the sum rule

$$\dim(A) = \frac{1}{\dim(U)} \sum_{\kappa\in\mathcal{J}} \dim(\dot{M}_\kappa)\,\langle U,\dot{M}_\kappa\rangle \qquad (4.45)$$

[18] Via these maps one actually gets functors $\mathcal{C} \to \mathcal{C}_A$ (induction) and $\mathcal{C}_A \to \mathcal{C}$ (restriction). Induction is a right-adjoint functor to restriction, and for modular $\mathcal{C}$ both functors are exact, i.e. carry exact sequences of morphisms to exact sequences [36]. Moreover, even when $\mathcal{C}_A$ is a tensor category, the induction functor is not a tensor functor. In contrast, there are functors from $\mathcal{C}$ to the bimodule category ${}_A\mathcal{C}_A$ (so-called α-induction, see section 5.4 below) which are tensor functors.

holds.

Proof:
(i) Formula (4.44) follows immediately by applying the reciprocity relation (4.43) to the simple A-modules M_κ.
(ii) Since dimensions are constant on isomorphism classes of objects, we can restrict our attention to $U = U_j$ for $j \in \mathcal{I}$. For these objects the result follows by taking the trace of both sides of the relation (4.44). ✓

In the vector space case the relation analogous to (4.45) reads $\dim(A) = \sum_\kappa (\dim(V_\kappa))^2$. Thus when this formula is extended to general tensor categories, the two factors of $\dim(V_\kappa)$ get generalised in two distinct ways, one of them to $\dim(\dot{M}_\kappa)/\dim(U_i)$, the other to $\langle U_i, \dot{M}_\kappa\rangle$. Let us also mention that for simple current algebras, these two factors take the form $\dim(\dot{M}_\kappa)/\dim(U_i)$ $= |G|/\sqrt{s_i u_i}$ and $\langle U_i, M_\kappa\rangle = \sqrt{s_i/u_i}$, respectively, where G is the simple current group, s_i is the order of the stabiliser of U_i (the subgroup of G of simple currents that leave i fixed), and u_i the order of the so-called untwisted stabiliser (for its definition see [105, 106]).

We are now also in a position to make the following assertion that will be useful later on (see section 5.7):

Theorem 4.15:
Let X, Y be simple objects of $\mathcal{C}$ and A a symmetric special Frobenius algebra in $\mathcal{C}$. Then the prescription (4.24) furnishes a natural isomorphism

$$\Psi\ :\quad \bigoplus_{\kappa\in\mathcal{J}} \mathrm{Hom}(X, \dot{M}_\kappa) \otimes \mathrm{Hom}(\dot{M}_\kappa, Y) \to \mathrm{Hom}(A\otimes X, Y)\,. \tag{4.46}$$

Proof:
Recall that the direct sum in (4.46) is over one representative $\dot{M}_\kappa$ out of each isomorphism class of simple A-modules. In the following for simplicity we will just write κ in place of $\dot{M}_\kappa$. To show injectivity, we apply Ψ to a basis of the space on the left hand side as in (4.25). Suppose that

$$\sum_{\kappa\in\mathcal{J}} \sum_{i,j\in\mathcal{I}} \sum_{\alpha,\beta} \lambda_{\kappa(i,\alpha)}^{(j,\beta)}\, \rho_{\kappa(i,\alpha)}^{(j,\beta)} = 0 \tag{4.47}$$

for some complex numbers $\lambda_{\kappa(i\alpha)}^{(j\beta)}$. Upon composition with another representation function $\rho_{\kappa'(i',\alpha')}^{(j',\beta')}$ and averaging over A, the left hand side yields $\lambda_{\kappa'(i',\alpha')}^{(j',\beta')}$ by lemma 4.9. Hence all the numbers $\lambda_{\kappa(i\alpha)}^{(j\beta)}$ are zero.
Surjectivity is checked as follows. Using the additivity of the ith dimension morphism (4.27), it follows from corollary 4.14 that

$$\begin{aligned}\mathcal{D}_i(\mathrm{Ind}_A(U_j)) &= \textstyle\sum_{\kappa\in\mathcal{J}} \langle U_j, \dot{M}_\kappa\rangle\, \mathcal{D}_i(M_\kappa) = \sum_{\kappa\in\mathcal{J}} \langle U_j, \dot{M}_\kappa\rangle\, \langle \dot{M}_\kappa, U_i\rangle\, \mathrm{id}_{U_i} \\ &\equiv \dim\big[\textstyle\bigoplus_{\kappa\in\mathcal{J}} \mathrm{Hom}(U_j, \dot{M}_\kappa) \otimes \mathrm{Hom}(\dot{M}_\kappa, U_i)\big]\, id_{U_i}\,.\end{aligned} \tag{4.48}$$

On the other hand, using directly formula (4.38) we get

$$\mathcal{D}_i(\mathrm{Ind}_A(U_j)) = \langle U_i, A\otimes U_j\rangle\, id_{U_i} \equiv \dim\big[\mathrm{Hom}(A\otimes U_j, U_i)\big]\, id_{U_i}\,. \tag{4.49}$$

Comparison of (4.49) and (4.48) establishes surjectivity of the map (4.46) for the case that $X = U_j$ and $Y = U_i$. By additivity this extends to the case of arbitrary objects X and Y (using also that $\mathcal{C}$ is semisimple). ✓

In the special case that A is a group algebra in Vect($\mathbb{C}$), the theorem amounts to the statement that for complex semisimple algebras the representation functions span the dual space of the algebra.

An interesting consequence of the relation (4.33) is that it allows us to invert the isomorphism (4.46) explicitly. We find:

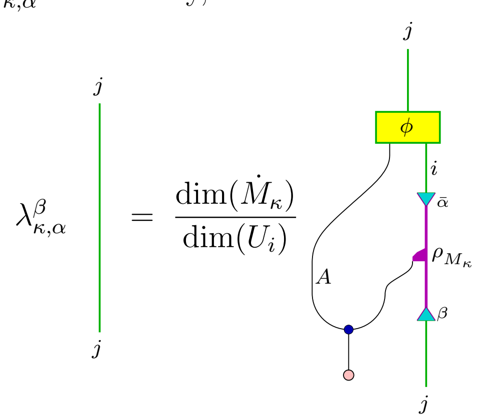
Lemma 4.16:
The coefficients $\lambda_{\kappa,\alpha}^{\ \beta}$ in the expansion

$$\phi = \sum_{\kappa\in\mathcal{J}} \sum_{\alpha,\beta} \lambda_{\kappa,\alpha}^{\ \beta}\, \rho_{M_\kappa(i,\alpha)}^{(j,\beta)} \tag{4.50}$$

of a morphism $\phi \in \mathrm{Hom}(A\otimes U_i, U_j)$ with respect to the representation functions (4.25) read

$$\lambda_{\kappa,\alpha}^{\ \beta}\, id_j = \frac{\dim(\dot{M}_\kappa)}{\dim(U_i)}\, \phi \circ [id_A \otimes (b_{M_\kappa}^{(i,\alpha)} \circ \rho_{M_\kappa})] \circ [(\Delta\circ\eta)\otimes b_{(j,\beta)}^{M_\kappa}]\,. \tag{4.51}$$

Proof:
Since the morphisms $\rho_{M_\kappa(i,\alpha)}^{(j,\beta)}$ form a basis of $\mathrm{Hom}(A\otimes U_i, U_j)$, there exists a unique set of numbers $\lambda_{\kappa,\alpha}^{\ \beta}$ with property (4.50). Their values can be extracted by composing the right hand side of (4.50) with a representation function and 'integrating' over A. In this way one arrives at the expression (4.51) for $\lambda_{\kappa,\alpha}^{\ \beta}$. Pictorially,



(4.52)

✓

4.4 From boundary conditions to representations

In section 3.2 we have seen how a boundary condition (preserving the chiral algebra $\mathfrak{V}$) in a rational CFT gives rise to a symmetric special Frobenius algebra A in the modular tensor category of the CFT (the representation category of $\mathfrak{V}$). Let us henceforth also label the boundary condition from which this algebra arises by A. Recall that the OPE of two boundary fields living on the A-boundary furnishes the multiplication of the algebra A, and the sewing constraint for four boundary fields on a disk assures its associativity.

The notion of an A-module has a direct interpretation in CFT terms as well. Consider a stretch of boundary for which at some point y the boundary condition changes from A to another boundary condition, say M. Then at the point y a 'boundary changing field' $\Psi^{AM}(y)$ must be inserted. The OPE of any field $\Phi(x)$ that lives on the boundary A with the boundary changing field $\Psi^{AM}(y)$ is forced to be of the form

$$\Phi(x)\,\Psi^{AM}(y) = \sum_k (x-y)^{\Delta_k-\Delta_\Phi-\Delta_\Psi}\,\Psi_k^{AM}(y)\,, \tag{4.53}$$

where Ψ_k^{AM} is a suitable collection of boundary changing fields. The boundary OPE thus defines a mapping

$$\left\{\begin{array}{c}\text{fields}\\ \text{living on } A\end{array}\right\} \times \left\{\begin{array}{c}\text{boundary changing}\\ \text{fields } A\to M\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{boundary changing}\\ \text{fields } A\to M\end{array}\right\}. \tag{4.54}$$

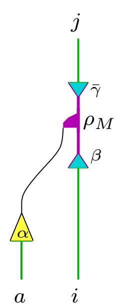
This is the analogue of the representation morphism ρ_M as defined in formula (4.1).

To make this qualitative description precise, let us write out the sewing constraint and the representation property in a basis and check whether they agree. We start with the general sewing constraint for four boundary changing fields on a disk. For two (not necessarily distinct) boundary conditions M,N, let $\Psi^{MN}_{a\alpha}(x)$ denote a boundary changing field [19] that is primary with respect to the chiral algebra $\mathfrak{V}$ and carries the representation a of $\mathfrak{V}$. The index α counts its multiplicity. For three boundary conditions M,N,K, let $C^{(MNK)c\gamma;\,\delta}_{a\alpha,b\beta}$ be the boundary structure constants appearing in the OPE $\Psi^{MN}_{a\alpha}(x)\,\Psi^{NK}_{b\beta}(y)$, defined by a direct generalisation of the OPE (3.11). Then the general sewing constraint reads [74, 14, 76]

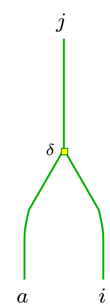
$$\sum_{\varphi=1}^{n_{fM}^{\ \ K}} C^{(MNK)f\varphi;\,\rho}_{a\alpha,b\beta}\, C^{(MKL)d\delta;\,\sigma}_{f\varphi,c\gamma} = \sum_{\varepsilon=1}^{n_{eN}^{\ \ L}} \sum_{e\in\mathcal{I}} \sum_{\varrho=1}^{N_{bc}^{\ e}} \sum_{\tau=1}^{N_{ae}^{\ d}} C^{(NKL)e\varepsilon;\,\varrho}_{b\beta,c\gamma}\, C^{(MNL)d\delta;\,\tau}_{a\alpha,e\varepsilon}\, \mathsf{F}^{\,(a\,b\,c)\,d}_{\tau e\varrho,\rho f\sigma}\,. \tag{4.55}$$

Here $n_{fM}^{\ \ K}$ gives the multiplicity of the boundary changing field $\Psi^{MK}_{f\varphi}$, and similarly for $n_{eN}^{\ \ L}$. In fact, these are nothing but the annulus coefficients (see section 5.8), $n_{iM}^{\ \ N} = \mathrm{A}_{iM}^{\ \ N}$.

Let us now write out the representation property (4.1) in a basis. To do so define the numbers



$$=: \sum_{\delta=1}^{N_{ai}^{\ j}} \rho^{M\,(j\gamma);\delta}_{(a\alpha)(i\beta)} \tag{4.56}$$



In terms of these the representation property reads

$$\sum_{\varphi=1}^{\langle f,\dot M\rangle} m^{f\varphi;\,\rho}_{a\alpha,b\beta}\, \rho^{M\,(d\delta);\sigma}_{(f\varphi)(c\gamma)} = \sum_{e\in\mathcal{I}} \sum_{\varepsilon=1}^{\langle e,\dot M\rangle} \sum_{\varrho=1}^{N_{bc}^{\ e}} \sum_{\tau=1}^{N_{ae}^{\ d}} \rho^{M\,(e\varepsilon);\varrho}_{(b\beta)(c\gamma)}\, \rho^{M\,(d\delta);\tau}_{(a\alpha)(e\varepsilon)}\, \mathsf{F}^{\,(a\,b\,c)\,d}_{\tau e\varrho,\rho f\sigma}\,. \tag{4.57}$$

[19] The boundary fields living on a given boundary M are just those fields that leave the boundary condition invariant. Thus in the present notation they are written as $\Psi^{MM}_{a\alpha}(x)$.

Thus we see that upon setting

$$C^{(AAA)c\gamma;\,\delta}_{a\alpha,b\beta} = m^{c\gamma;\,\delta}_{a\alpha,b\beta} \qquad \text{and} \qquad C^{(AAM)j\gamma;\,\delta}_{a\alpha,i\beta} = \rho^{M\,(j\gamma);\delta}_{(a\alpha)(i\beta)}\,, \tag{4.58}$$

formula (4.57) becomes the special case $M = N = K = A$, $L = M$ of (4.55). As an object in $\mathcal{C}$, the module defined by the boundary condition M is given by

$$\dot M \cong \bigoplus_{i\in\mathcal{I}} \bigoplus_{\alpha=1}^{n_{iA}{}^{M}} U_i\,, \qquad \text{i.e.} \quad \langle i,\dot M\rangle = n_{iA}{}^{M}\,. \tag{4.59}$$

Thus the module M consists of all representations of the chiral algebra $\mathfrak{V}$ that occur as boundary changing fields $A \to M$, counted with multiplicities.

Together with the results of section 3.2 we have now learned the following. Fix an arbitrary boundary condition A; this defines an algebra object. Moreover, every boundary condition (including A itself) defines a module over this algebra. In other words, in our approach *boundary conditions are labelled by modules of the algebra object A*. It also turns out that elementary boundary conditions correspond to simple A-modules, see section 5.8 below. Direct sums of simple modules, in contrast, correspond to Chan–Paton multiplicities, which play an important role in type I string theories (for a recent review see [107]). It will be shown in a forthcoming paper that with the help of the data provided by the algebra A one can indeed define structure constants that solve the sewing constraint (4.55) in full generality (not merely in the special cases (3.14) and (4.58)).

There is no intrinsic meaning to the statement that boundary conditions in a CFT correspond to *left* rather than to *right* A-modules. This is just a convention we chose, and everything could as well be formulated in terms of right modules. Given a left A-module $M = (\dot M, \rho_M)$ we can construct a right A-module $(\dot M^\vee, f(\rho_M))$, where $f(\rho_M) \in \mathrm{Hom}(\dot M^\vee \otimes A, \dot M^\vee)$ is given by

$$f(\rho_M) = (d_{\dot M} \otimes id_{\dot M^\vee}) \circ (id_{\dot M^\vee} \otimes \rho_M \otimes id_{\dot M^\vee}) \circ (id_{\dot M^\vee} \otimes id_A \otimes b_{\dot M})\,. \tag{4.60}$$

It is easy to check that the mapping $(\dot M, \rho_M) \mapsto (\dot M^\vee, f(\rho_M))$ is invertible and that $f(\rho_M)$ is a right representation morphism (as defined below (4.2)). The right module $(\dot M^\vee, f(\rho_M))$ describes the same boundary condition as the left module $(\dot M, \rho_M)$.

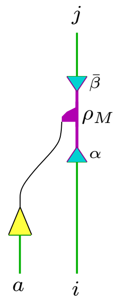
The notion of an A-B-bimodule, given in definition 4.5, also has a CFT interpretation, which we mention briefly here (it will be studied in more detail elsewhere). For $A = B$, the bimodules correspond to generalised disorder lines, a notion introduced in [34] and studied further in [16, 108]. For $A \neq B$, bimodules describe tensionless interfaces between two CFTs. The CFTs on each side of the defect are the ones obtained from the algebra objects A and B, respectively. Thus they contain the same chiral algebra $\mathfrak{V}$ and in particular have the same Virasoro central charge. These defects are special cases of those considered in [109, 110], where they are treated as boundary states in the product [20] of the two CFTs (this is known as the folding trick, see e.g. [111, 112]).

[20] This product must not be confused with the CFT associated to the tensor product of two algebras, see the discussion at the beginning of section 3.5.

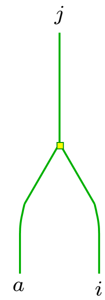
4.5 The case ${N_{ij}}^k \in \{0,1\}$ and $\dim \mathrm{Hom}(U_k, A) \in \{0,1\}$

In this section we study the representation theory of A under the simplifying assumptions that ${N_{ij}}^k \in \{0,1\}$ for all $i,j,k \in \mathcal{I}$ and that $\dim \mathrm{Hom}(k,A) \in \{0,1\}$ for all $k \in \mathcal{I}$. In contrast, we do *not* restrict our attention to such modules which contain simple objects only with multiplicity zero or one.

The notation $\rho^{M\,(j\gamma);\delta}_{(a\alpha)(i\beta)}$ introduced in equation (4.56) to describe the representation morphism in a basis now reduces to $\rho^{M\,j\gamma}_{a\,,\,i\beta}$, i.e.



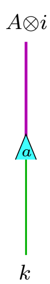
$$= \quad \rho^{M\,j\beta}_{a\,,\,i\alpha} \tag{4.61}$$



When expressed in terms of these numbers, the representation properties (4.1) also take an easier form than in formula (4.57):

$$\rho^{M\,j\beta}_{0\,,\,i\alpha} = \delta_{i,j}\,\delta_{\alpha,\beta} \qquad \text{and} \qquad \sum_{(k\gamma)\prec M} \rho^{M\,j\beta}_{a\,,\,k\gamma}\,\rho^{M\,k\gamma}_{b\,,\,i\alpha}\,\mathsf{F}^{(a\,b\,i)\,j}_{k\,c} = \delta_{c\prec A}\,{m_{ab}}^c\,\rho^{M\,j\beta}_{c\,,\,i\alpha}\,. \tag{4.62}$$

Solving this polynomial equation is not easy. But fortunately we can invoke proposition 4.13, which states that every irreducible module is a submodule of an induced module. Therefore, instead of attacking (4.62) directly, we start by working out the matrix $\rho^{A\otimes i\,\ell\beta}_{a\,,\,k\alpha}$ of an induced module $\mathrm{Ind}_A(U_i)$. We choose the bases



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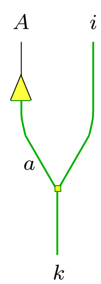
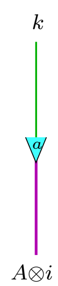


and



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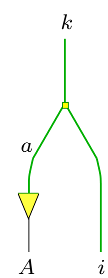
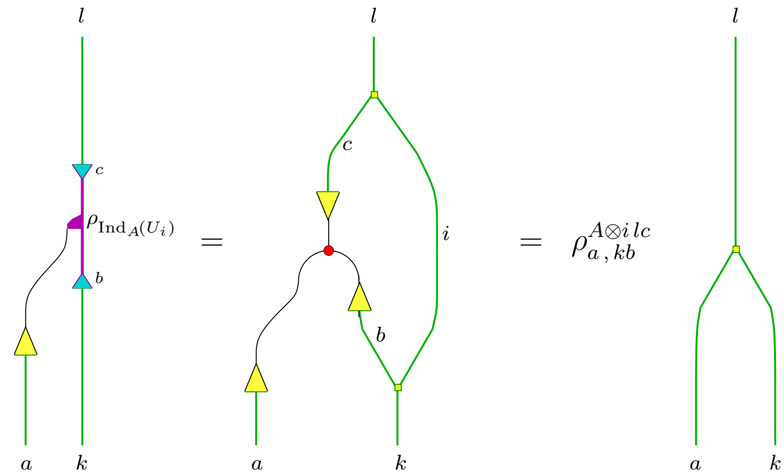


(4.63)

in $\mathrm{Hom}(U_k, A\otimes U_i)$ and $\mathrm{Hom}(A\otimes U_i, U_k)$; they are dual to each other in the sense of (4.21). The basis vectors are labelled by those $a \prec A$ that occur in the fusion $U_k \otimes U_{\bar\imath}$. The set is complete, since $\dim \mathrm{Hom}(U_k, A\otimes U_i) = \dim \mathrm{Hom}(U_k\otimes U_{\bar\imath}, A)$.

To find the representation matrix (4.61) in this basis, one must evaluate



(4.64)

Using the definition of G (2.40), this leads to the simple formula

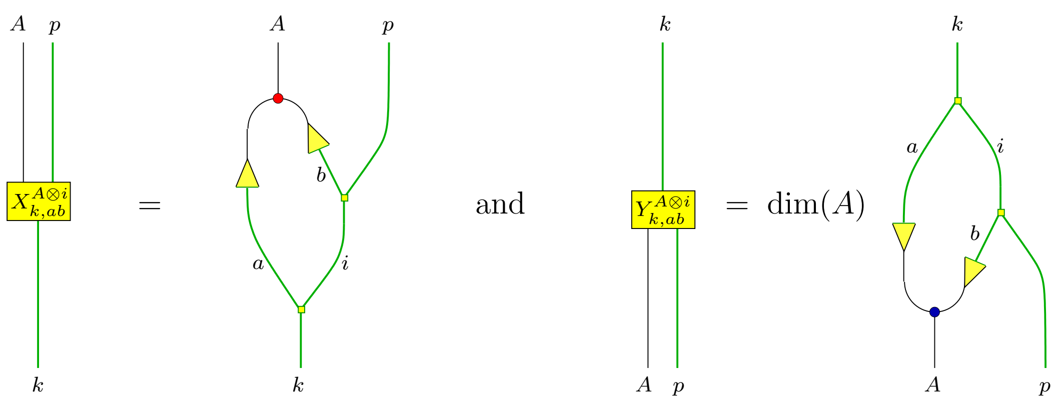
$$\rho^{A\otimes i\,lc}_{a\,,\,kb} = m_{ab}{}^{c}\,\mathsf{G}^{(a\,b\,i)\,l}_{c\,k}\,. \tag{4.65}$$

In case the induced module $\mathrm{Ind}_A(U_i)$ is a simple module, then we are already done. If it is not, then we must find a projector decomposition of the space $\mathrm{Hom}_A(\mathrm{Ind}_A(U_i),\mathrm{Ind}_A(U_i))$ to extract the simple submodules. This may require some work, but it amounts to solving *linear* equations only.

Suppose now that a simple module M that is not itself an induced module occurs as a submodule in

$$\mathrm{Ind}_A(U_p) \;\cong\; \mathrm{Ind}_A(U_{i_1}) \oplus \cdots \oplus \mathrm{Ind}_A(U_{i_n}) \oplus M\,, \tag{4.66}$$

with all induced modules $\mathrm{Ind}_A(U_{i_m})$ being simple.[21] Given (4.66), we can extract the representation matrix ρ^M with a little extra work from our knowledge of the induced representations. To this end, we choose a special basis in the spaces $\mathrm{Hom}(U_k, A{\otimes}U_p)$ together with a dual basis in $\mathrm{Hom}(A{\otimes}U_p, U_k)$. We do so in two steps. First, define the morphisms



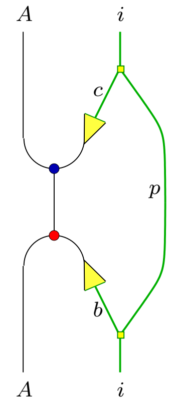
(4.67)

These are indeed dual to each other,

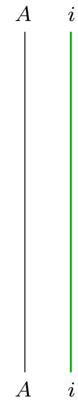
$$Y^{A\otimes j}_{k,cd} \circ X^{A\otimes i}_{k,ab} = \delta_{i,j}\,\delta_{a,c}\,\delta_{b,d}\,\mathrm{id}_{U_k}\,. \tag{4.68}$$

[21] This is of course not the generic situation. In particular, it may well happen that two unknown simple modules M and N always occur as a pair $M\oplus N$, in which case one cannot escape the full projector calculation. But the special situation studied here is e.g. realised in the E_7 example that we will investigate below.

This follows from the fact that $\dim \mathrm{Hom}_A(\mathrm{Ind}_A(U_i), \mathrm{Ind}_A(U_j)) = \delta_{i,j}$ when $\mathrm{Ind}_A(U_i)$ and $\mathrm{Ind}_A(U_j)$ are simple A-modules. In particular, the morphism on the left hand side of



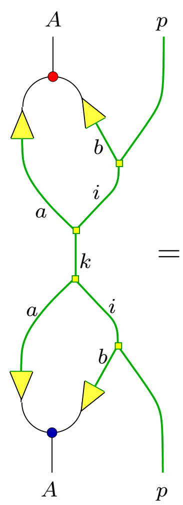
$$= \xi_{bc}$$



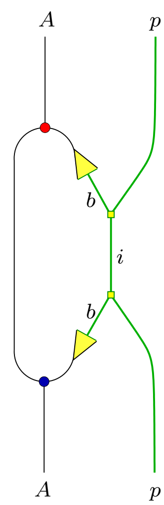
(4.69)

must be a scalar multiple of $id_{A\otimes i}$. Composing both sides of this relation with unit and counit, one determines $\xi_{bc} = \delta_{b,c}/\dim(A)$. Another important property of the morphisms X, Y is that they allow us to build elements in $\mathrm{Hom}_A(\mathrm{Ind}_A(U_p), \mathrm{Ind}_A(U_p))$:

$$\sum_{k,a\in\mathcal{I}} X^{A\otimes i}_{k,ab} \circ Y^{A\otimes i}_{k,ab} \;=\; \sum_{k,a\in\mathcal{I}}$$



$$= \dim(A)$$



$$\in \mathrm{Hom}_A(\mathrm{Ind}_A(U_p), \mathrm{Ind}_A(U_p)) \qquad (4.70)$$

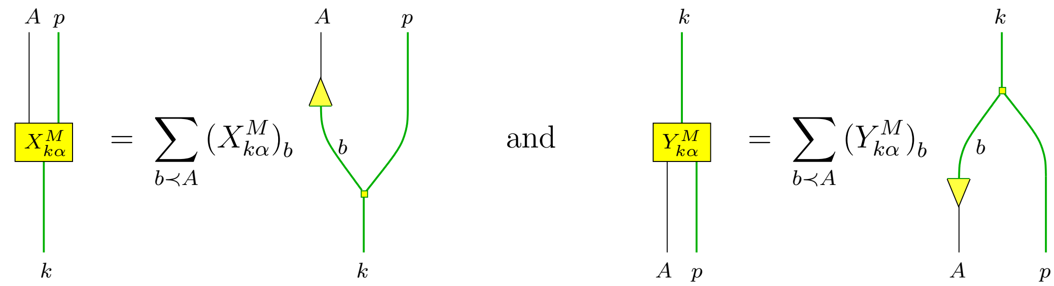
The morphisms X are linearly independent, but do not yet form a basis for all of $\mathrm{Hom}(U_k, A\otimes U_p)$, just because the module M is still missing. So we must find elements

$$X^M_{k\alpha} \in \mathrm{Hom}(U_k, A\otimes U_p) \qquad \text{and} \qquad Y^M_{k\alpha} \in \mathrm{Hom}(A\otimes U_p, U_k) \tag{4.71}$$

such that we have a basis in each of these spaces, and such that the Ys are still dual to the Xs. Using this basis, we can apply dominance in the form

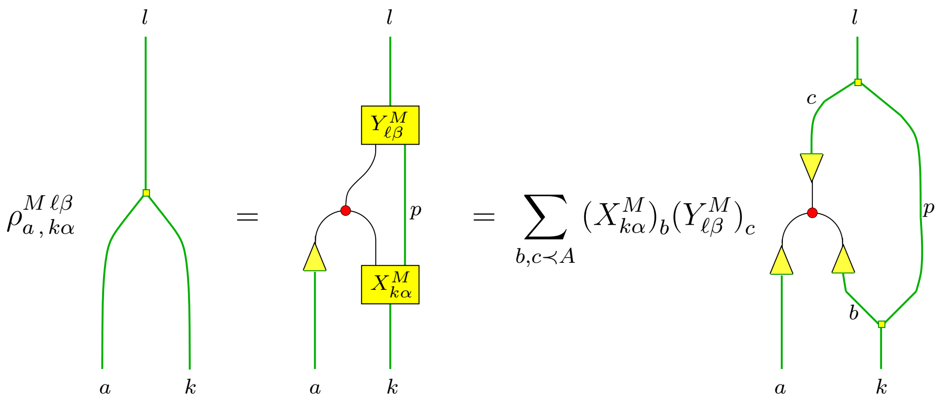
$$id_{A\otimes U_p} = \sum_{i,k\in\mathcal{I}} \sum_{a,b\prec A} X^{A\otimes i}_{k,ab} \circ Y^{A\otimes i}_{k,ab} + \sum_{k\in\mathcal{I}} \sum_{\alpha} X^M_{k\alpha} \circ Y^M_{k\alpha} \,. \tag{4.72}$$

Now since both the left hand side and the first term of the right hand side of this formula are in Hom_A, the last term is in Hom_A as well. To finally obtain the representation matrix ρ^M, express X^M and Y^M in a basis as



(4.73)

The relation



(4.74)

then determines ρ^M to be

$$\rho^{M\,\ell\beta}_{a\,,\,k\alpha} = \sum_{b,c} (X^M_{k\alpha})_b\,(Y^M_{\ell\beta})_c\,m_{ab}{}^c\,\mathsf{G}^{\,(a\,b\,p)\,\ell}_{\,c\,k}\,. \tag{4.75}$$

(Recall that the label p refers to the induced module $\mathrm{Ind}_A(U_p)$ form which the simple submodule M has been extracted.)

4.5.1 Example: Free boson

For the free boson, all induced modules turn out to be simple. (Thus the second half of the arguments above is not needed). To see this, recall the definition of the algebra object A_{2r} in formula (3.84). The dimensions of the morphism spaces Hom_A are then given by

$$\dim \mathrm{Hom}_A(\mathrm{Ind}_{A_{2r}}([n]), \mathrm{Ind}_{A_{2r}}([m])) = \dim \mathrm{Hom}([n], A_{2r}{\otimes}[m]) = \begin{cases} 1 & \text{if } n{-}m = 0 \bmod 2r\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.76}$$

In particular, indeed every induced module is simple. But not all of them are distinct. Rather, we have

$$\mathrm{Ind}_{A_{2r}}([n]) \cong \mathrm{Ind}_{A_{2r}}([n{+}2r]) \tag{4.77}$$

for any n (and no further isomorphisms). A set of representatives for each equivalence class of simple modules is thus given by

$$M_k = \mathrm{Ind}_{A_{2r}}([k]) \qquad \text{for} \quad k = 0, 1, \dots, 2r{-}1\,. \tag{4.78}$$

The multiplication can be read off the formula for induced modules (4.65):

$$\rho^{M_n\,[k+a]}_{[a]\,,\,[k]} = m_{[a][k-n]}{}^{[a+k-n]}\,\mathsf{G}^{([a]\,[k-n]\,[n])\,[k+a]}_{[a+k-n]\,[k]} = 1 \tag{4.79}$$

(and 0 else). In deriving this formula one uses that $a \prec A$ is even and that $k-n \geq 0$ is a multiple of $2r$ and thus is even, too. Note that the $1{\times}1$-matrix F in (2.77) is just a sign and hence equals its own inverse, $\mathsf{G} = \mathsf{F}$.

4.5.2 Example: E_7 modular invariant

For the E_7 modular invariant we need to determine the representation theory of the algebra object $A = (0) \oplus (8) \oplus (16)$ in the modular category formed by the integrable highest weight representations of $\mathfrak{su}(2)_{16}$.

As a first step we work out the embedding structure of the induced modules. The following table gives $\dim \mathrm{Hom}_A(\mathrm{Ind}_A(i), \mathrm{Ind}_A(j))$ for $i, j = 0, 1, \ldots, 8$. (For easier reading, in the table zero entries are indicated by a dot.)

$$\begin{array}{c|ccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
2 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
3 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & \cdot & 1 \\
5 & \cdot & \cdot & \cdot & 1 & \cdot & 2 & \cdot & 1 & \cdot \\
6 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot & 1 \\
7 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 2 & \cdot \\
8 & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 3
\end{array} \tag{4.80}$$

All other cases follow from the table via $\mathrm{Ind}_A(r) \cong \mathrm{Ind}_A(16{-}r)$, an isomorphism due to the simple current (16). From the diagonal entries of (4.80) we see that the four induced modules

$$\mathrm{Ind}_A(0)\,, \quad \mathrm{Ind}_A(1)\,, \quad \mathrm{Ind}_A(2)\,, \quad \mathrm{Ind}_A(3)\,. \tag{4.81}$$

are simple, while the other induced modules decompose as

$$\begin{array}{lll}
\mathrm{Ind}_A(4) \cong P \oplus Q\,, & \mathrm{Ind}_A(5) \cong \mathrm{Ind}_A(3) \oplus R\,, & \mathrm{Ind}_A(6) \cong \mathrm{Ind}_A(2) \oplus Q\,, \\
\mathrm{Ind}_A(7) \cong \mathrm{Ind}_A(1) \oplus \mathrm{Ind}_A(3)\,, & \mathrm{Ind}_A(8) \cong \mathrm{Ind}_A(0) \oplus \mathrm{Ind}_A(2) \oplus P\,, &
\end{array} \tag{4.82}$$

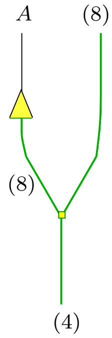
with simple A-modules P, Q, R that are not isomorphic to any induced module. As objects in $\mathcal{C}$, these simple modules decompose into simple objects as

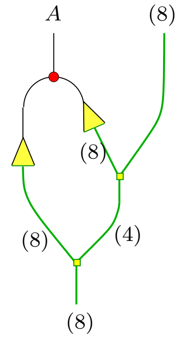
$$\dot{P} \cong (4) \oplus (8) \oplus (12)\,, \qquad \dot{Q} \cong (4) \oplus (6) \oplus (10) \oplus (12)\,, \qquad \dot{R} \cong (5) \oplus (11)\,. \tag{4.83}$$

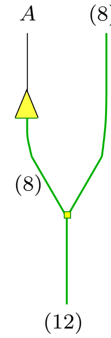
In agreement with the discussion above we can extract P from $\mathrm{Ind}_A(8)$, Q from $\mathrm{Ind}_A(6)$ and R from $\mathrm{Ind}_A(5)$. Let us start with $\mathrm{Ind}_A(8)$. Its decomposition into simple objects of $\mathcal{C}$ reads

$$\begin{array}{rllll}
A{\otimes}(8) \;\cong & (0) & \oplus\,(8)\,\oplus & (16) & [\,\text{from } A{\otimes}(0)\,] \\
\oplus & (2) & \oplus\,(6) \oplus (8) \oplus (10)\,\oplus & (14) & [\,\text{from } A{\otimes}(2)\,] \\
\oplus & (4) & \oplus\,(8)\,\oplus & (12) & [\,\text{from } P\,]
\end{array} \tag{4.84}$$

One verifies that to complete the bases in the spaces $\mathrm{Hom}((4), A{\otimes}(8))$, $\mathrm{Hom}((8), A{\otimes}(8))$ and $\mathrm{Hom}((12), A{\otimes}(8))$ one may choose the elements

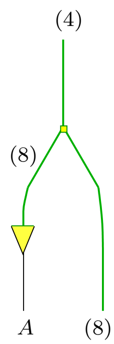
$X^P_{(4)} =$ 

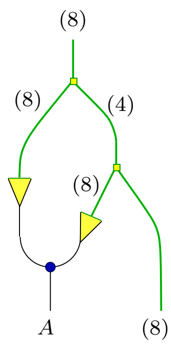
$X^P_{(8)} = \xi^{-1}$ 

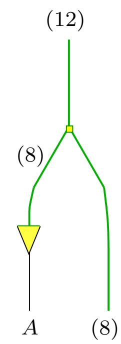
$X^P_{(12)} =$ 

(4.85)

where the normalisation constant ξ is defined implicitly by the condition $Y^P_{(8)} \circ X^P_{(8)} = id_{(8)}$, and the dual basis elements are

$Y^P_{(4)} =$ 

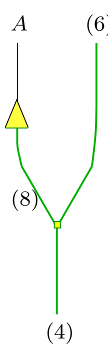
$Y^P_{(8)} =$ 

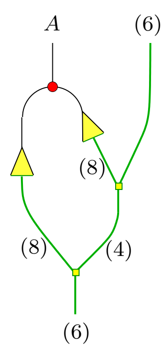
$Y^P_{(12)} =$ 

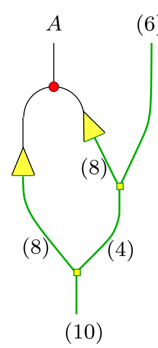
(4.86)

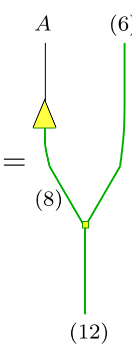
The only non-trivial case is $X^P_{(8)}$. The calculation that this indeed extends $X^{A\otimes(0)}_{(8)}$ and $X^{A\otimes(2)}_{(8)}$ (in the conventions defined in (4.67)) to a basis of $\mathrm{Hom}((8), A{\otimes}(8))$ with dual basis given by $Y^{A\otimes(0)}_{(8)}$, $Y^{A\otimes(2)}_{(8)}$ and $Y^P_{(8)}$ amounts to the observation that $\dim \mathrm{Hom}_A(\mathrm{Ind}_A(4), \mathrm{Ind}_A(r)) = 0$ for $r = 0, 2$.

The same procedure works to extract Q from $\mathrm{Ind}_A(6)$, giving the additional basis elements

$X^Q_{(4)} =$ 

$\xi_1\, X^Q_{(6)} =$ 

$\xi_2\, X^Q_{(10)} =$ 

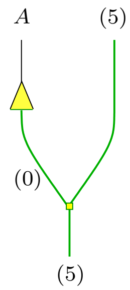
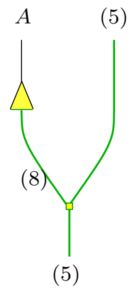
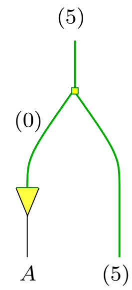
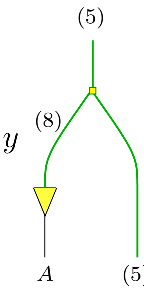
$X^Q_{(12)} =$ 

(4.87)

with duals defined in the same manner as in (4.86) and ξ_1, ξ_2 defined implicitly by the conditions $Y^Q_{(6)} \circ X^Q_{(6)} = id_{(6)}$ and $Y^Q_{(10)} \circ X^Q_{(10)} = id_{(10)}$.

For the module R this short-cut does not work, and we need to solve a linear system to complete the basis and its dual. But since both morphism spaces (embedding (5) or (11) into

$A\otimes(5)$) are two-dimensional, we can easily write down (one choice for) the missing vectors:

$X^R_{(5)} \;=\; u$  $+\; v$  $\qquad Y^R_{(5)} \;=\; x$  $+\; y$  $\qquad (4.88)$

where, in the notation (4.73) from above,

$$u = (Y^{A\otimes(3)}_{(5)})_{(8)}\,, \quad v = -(Y^{A\otimes(3)}_{(5)})_{(0)}\,, \quad x = (X^{A\otimes(3)}_{(5)})_{(8)}\,, \quad y = -(X^{A\otimes(3)}_{(5)})_{(0)}\,. \tag{4.89}$$

The morphisms $X^R_{(11)}$ and $Y^R_{(11)}$ can be determined in the same way.

We have now gathered the necessary information to give the representation matrix of the simple A-modules P, Q, R via equation (4.75). As an additional check of the result, we verified numerically for a random sample of cases that these representation matrices indeed solve the non-linear constraint (4.62) for $\mathfrak{su}(2)_{16}$.

5 Partition functions

Before proceeding, let us recall a few aspects of our general philosophy. The construction we want to implement can be summarised as

$$\boxed{\text{chiral data}} + \boxed{\text{symm. special Frobenius algebra}} \longrightarrow \boxed{\text{full CFT}} \tag{5.1}$$

By chiral data we mean the representation theory of the chiral algebra and the conformal blocks of a rational conformal field theory. The corresponding modular category contains strictly less information than the chiral data. Roughly speaking, the category encodes only the monodromies of the blocks, but not their functional dependence on the insertion points and the moduli of the world sheet or the information which state of a given representation of the chiral algebra is inserted.[22] But it turns out that much important information about a CFT, like its field content, its boundary conditions and defect lines, the OPE structure constants, and the consistency of these data with factorisation, can be discussed entirely at the level of the modular category. It is a strength of the present approach that, in a sense, in dealing with the modular category $\mathcal{C}$ one 'forgets' exactly the right amount of complexity of the chiral data to render those problems tractable.

In order to select which full CFT is to be constructed from a given set of chiral data, additional input is required. In our approach this input is a symmetric special Frobenius algebra object A in $\mathcal{C}$. The full CFT that one obtains this way – the one indicated on the right hand side of (5.1) – will be specified in terms of its correlation functions. To this end, the correlators on an arbitrary world sheet X are expressed as specific elements in the vector spaces of conformal blocks on the complex double $\hat{\mathrm{X}}$ of X. Such an element is described by a ribbon graph in a three-manifold M_{X}, called the *connecting three-manifold*, whose boundary ∂M_{X} is equal to the double $\hat{\mathrm{X}}$. Which three-manifold M_{X} is to be used, and which ribbon graph in M_{X}, was established for the Cardy case in [113, 51] (for all possible world sheets), and described in the general case in [22] (with restriction to orientable world sheets).

The idea to study a chiral CFT as the edge system of a (topological) theory on a three-dimensional manifold is quite natural and has been put forward in various guises, see for instance [114, 115, 116]. It finds a physical realisation in the CFT description of quantum Hall fluids in the scaling limit (see [117] for a review). By exploiting chiral CFT on the complex double $\hat{\mathrm{X}}$, this relationship can be used to study *full* CFT on the world sheet X.

In the following we present this construction in the form that it takes for orientable world sheets without field insertions. Afterwards we further specialise to the cases where the world sheet is a torus or an annulus.

[22] More precisely, the modular tensor category encodes the information about the conformal blocks as abstract vector bundles $\mathcal{V}$ over the moduli space (which for m insertion points labelled by $\vec{\imath} := (i_1, i_2, \dots, i_m) \in \mathcal{I}^m$ is the moduli space $\mathcal{M}_{g,m}$ of complex curves of genus g with m marked points). There are actually two different formulations of TFT: The topological modular functor works with vector spaces and provides a representation of the mapping class group. The complex modular functor gives a local system over the moduli space $\mathcal{M}_{m,g}$. The two descriptions are related by a Riemann–Hilbert problem, so that they contain equivalent information [39]. The full chiral data provide much more: They specify an embedding of $\mathcal{V}_{\vec{\imath}}$ in the trivial bundle over $\mathcal{M}_{g,m}$ whose fiber is the algebraic dual of the tensor product $\mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2} \otimes \cdots \otimes \mathcal{H}_{i_m}$ of the vector spaces underlying the irreducible representations of the chiral algebra. This information allows to obtain the values of conformal blocks on all chiral states.

5.1 The connecting manifold and the ribbon graph

Having fixed a modular category $\mathcal{C}$ and a symmetric special Frobenius algebra A in $\mathcal{C}$, to construct the correlator on a world sheet without field insertions we must in addition provide the following data:

- An orientable world sheet X, possibly with boundaries;
- an orientation on X;
- for each component of the boundary ∂X an A-module that specifies the boundary condition.

Here an important point is that orientability of the world sheet is not sufficient. Rather, to determine the correlator uniquely we need to select an actual *orientation*. For the correlator to be defined unambiguously without specifying the orientation, more structure than just a symmetric special Frobenius algebra is needed. This is linked to the fact that there exist CFTs which cannot be defined on non-orientable world sheets while still preserving the chiral algebra $\mathfrak{V}$, most notably those whose torus partition function is not symmetric, $Z_{ij} \neq Z_{ji}$. Since also such modular invariants can be described in terms of a Frobenius algebra, some *extra structure* is needed to ensure that the theory is consistent also on non-orientable world sheets. It turns out that the relevant extra structure is given by a *conjugation* on A, which furnishes an isomorphism between A and $A_{\rm op}$ and may be thought of as providing a square root of the twist. Not every symmetric special Frobenius algebra possesses such an extra structure and for those who do, the conjugation need not be unique. Moreover, the number of conjugations is *not* Morita invariant. We will return to this issue in a future publication.

The second important aspect of the prescription is that boundary conditions are labelled by A-modules. We will prove below that *simple* A-modules provide a complete set of *elementary* boundary conditions. This implies in particular that the number of elementary boundary conditions is the same as the number of Ishibashi states in the (dual of the) bulk state space. A general A-module is a direct sum of simple modules and corresponds to a superposition of elementary boundary conditions.

All these boundary conditions preserve the chiral algebra $\mathfrak{V}$ whose representation category is the modular category $\mathcal{C}$. Recall that $\mathfrak{V}$ is not necessarily the maximally extended chiral algebra – in the most extreme case, it is just the Virasoro algebra. As a consequence, our formalism covers symmetry breaking boundary conditions as well, as long as the subalgebra of the chiral algebra that is preserved by all boundary conditions under study is still rational.

The complex double $\hat{\rm X}$ of an orientable world sheet X can be obtained by taking two disjoint copies of X, with opposite orientation, and gluing them together along the boundary:

$$\hat{\rm X} := ({\rm X} \times \{-1,1\})/{\sim} \qquad \text{with} \quad (x,1) \sim (x,-1) \quad \text{iff} \quad x \in \partial{\rm X}\,. \tag{5.2}$$

The connecting three-manifold $M_{\rm X}$ is then the following natural interval bundle over X [113]:

$$M_{\rm X} := ({\rm X} \times [-1,1])/{\sim}\,; \tag{5.3}$$

the equivalence relation $\sim$ now identifies the values t and $-t$ in all intervals $\{x\} \times [-1,1]$ that lie above a boundary point of X, i.e.

$$(x,t) \sim (x,-t) \quad \text{for all} \quad x \in \partial{\rm X} \quad \text{and all} \quad t \in [-1,1]\,. \tag{5.4}$$

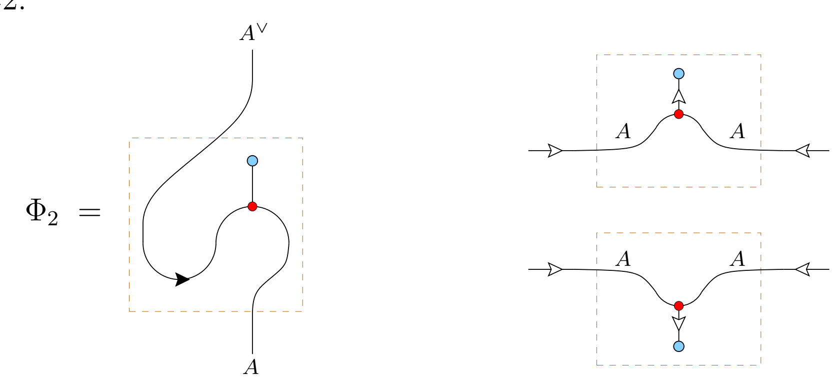
One quickly checks that in this manner one indeed achieves $\partial M_{\mathrm{X}} = \hat{\mathrm{X}}$. Moreover, the world sheet X is naturally embedded in M_{X}, via the map

$$\iota : \quad \mathrm{X} \to M_{\mathrm{X}} , \quad x \mapsto (x, 0) . \tag{5.5}$$

Thereby X is a retract of M_{X}, and conversely the connecting manifold M_{X} can be regarded as a fattening of the world sheet (and hence in particular its construction does not introduce any additional homotopical information). In the sequel we will always think of the world sheet X as being embedded in M_{X} in this fashion. Via the embedding (5.5) each boundary component of X coincides with a circular line of singular points of M_{X} that results from the fixed points under the identification (5.4).

We now describe how the ribbon graph in M_{X} is constructed. This construction involves several choices. As we will show afterwards, the invariant associated to the graph is independent of all these choices.

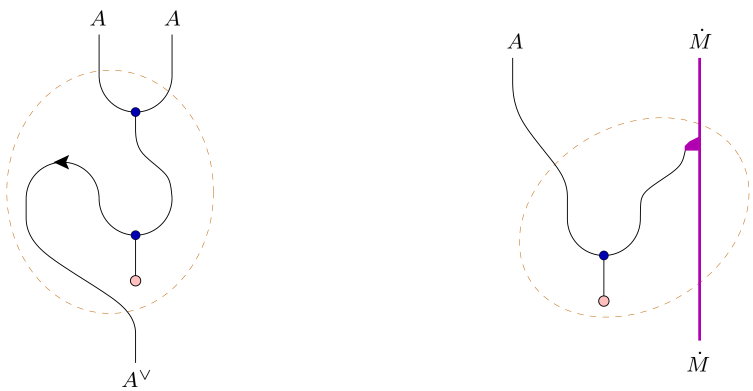
- First choose a triangulation of the world sheet X – choice #1. To be precise, the faces of the 'triangulation' are allowed to possess arbitrarily many edges, while all the vertices are required to be trivalent (so strictly speaking, this is the dual of a genuine triangulation).[23]
- All ribbons lie in the two-dimensional submanifold X of M_{X}. Furthermore, all ribbons (regarded as embedded two-manifolds) are to be oriented in such a way that their orientation coincides with the one induced from X – in short, the "white" side of each ribbon faces "up". The boundary of X is taken to be oriented such that upon an orientation preserving map of a patch of X that contains a boundary segment to the upper half plane the orientation of the image of the segment agrees with that of the real axis.
- For every boundary component of X, labelled by an A-module M, place an annular M-ribbon in $\mathrm{X} \subset M_{\mathrm{X}}$, along the circular line of singular points of M_{X} that corresponds to the boundary component. The orientation of the core of the ribbon must be taken to agree with the orientation of the boundary component.
- On those edges of the triangulation that are not part of the boundary of X, place A-ribbons that are directed away from the vertices. In the middle of each edge these are to be joined by the morphism $\Phi_2 \in \mathrm{Hom}(A, A^\vee)$ that we defined in (3.32) and display once again on the left hand side of figure (5.6) below. As indicated on the right hand side of (5.6), this can be done in two distinct ways; pick one of them on each edge – this is choice #2.



(5.6)

[23] It is in fact sufficient to place an edge along each boundary component and along a basis of non-contractible cycles of the two-manifold (resolving any four-point vertices that result from this prescription into two three-point vertices). This amounts to the rule that the 'triangulation' needs to be just fine enough such that any further edge that is added can be removed by repeated use of the fusion and bubble moves given in (5.10) below.

- At every trivalent vertex in the interior of X, join the three outgoing A-lines with the morphism $\Delta \circ \Phi_1^{-1} \in \mathrm{Hom}(A^\vee, A\otimes A)$, see the left part of figure (5.7). This morphism can be inserted in three different ways – choice #3.
- At every trivalent vertex on a boundary component of X that is labelled by M, put the morphism $(id_A \otimes \rho_M) \circ (\Delta \otimes id_{\dot M}) \circ (\eta \otimes id_{\dot M}) \in \mathrm{Hom}(\dot M, A\otimes \dot M)$, with ρ_M the representation morphism of M, as in the right part of figure (5.7).

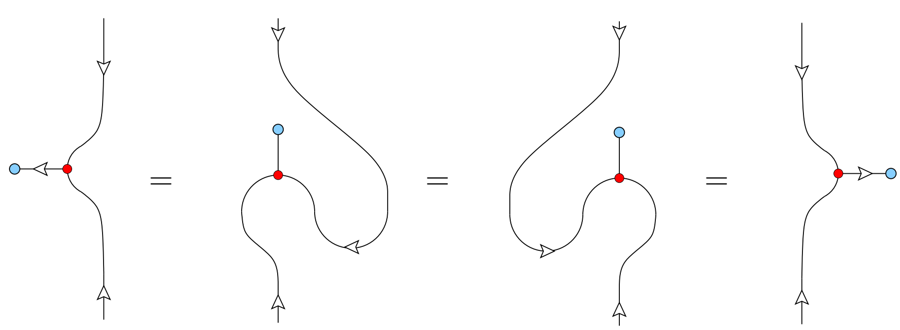


(5.7)

This construction will be illustrated for the torus and the annulus in sections 5.3 and 5.8, respectively.

According to the discussion in section 2.4, the invariant of the ribbon graph in M_{X} constructed as above is a vector in the space of conformal zero-point blocks on $\partial M_{\mathrm{X}} = \hat{\mathrm{X}}$, which is the value of the correlator on X. We proceed to show that this element is independent of the three choices made in the construction. We will call two ribbon graphs in a three-manifold *equivalent* if they possess the same invariant for any modular category $\mathcal{C}$. In this terminology, what we are going to show is that different choices always lead to equivalent ribbon graphs.

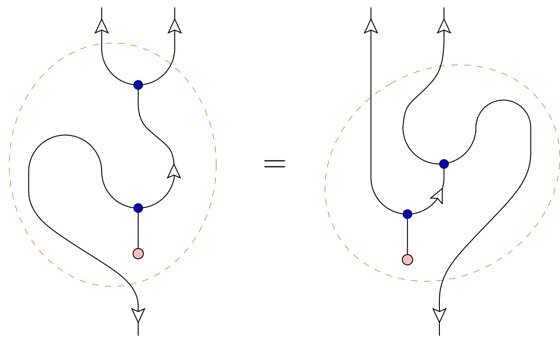
- Choice #2 – two ways to insert Φ_2: The two fragments of the ribbon graph can be transformed into each other as follows (all ribbons are A-ribbons):



(5.8)

The first equality is just a deformation of the ribbon graph, while the second equality amounts to the fact that the algebra A is symmetric, as in (3.33).

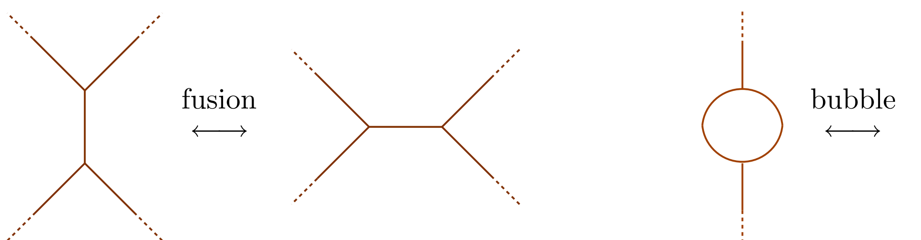
- Choice #3 – coupling three A-ribbons: It is sufficient to show that a 120° clockwise rotation of the encircled vertex that appears on the left hand side of (5.7) does produce equivalent ribbon graphs, i.e. that (all ribbons in the figure are A-ribbons)



(5.9)

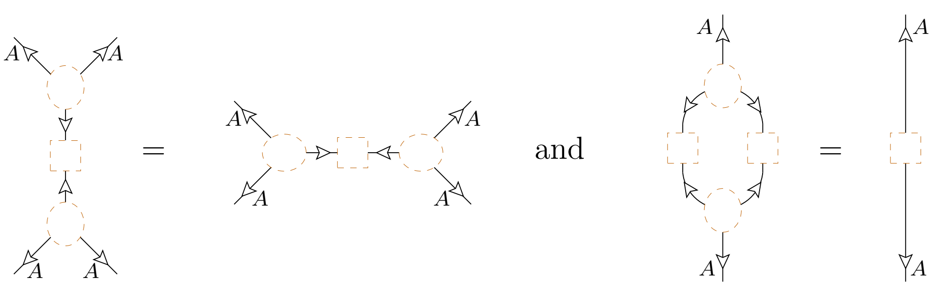
The left hand side of this equation can be transformed into the right hand side by first using that A is symmetric, i.e. that $\Phi_1^{-1} = \Phi_2^{-1}$ for the morphisms (3.35), on the lower coproduct, and then coassociativity as in (3.27).

- Choice #1 – triangulation of the world sheet: Any two triangulations of a Riemann surface can be transformed into each other via the so-called *fusion* and *bubble* moves (see e.g. [89, 66, 92]), which look as follows:



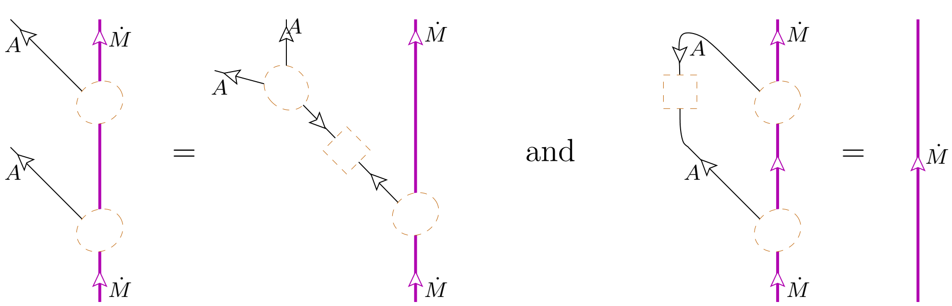
(5.10)

Two triangulations related by one of these moves give rise to equivalent ribbon graphs if the following transformations can be performed with the help of the properties of the algebra A and the module M: In the interior,



(5.11)

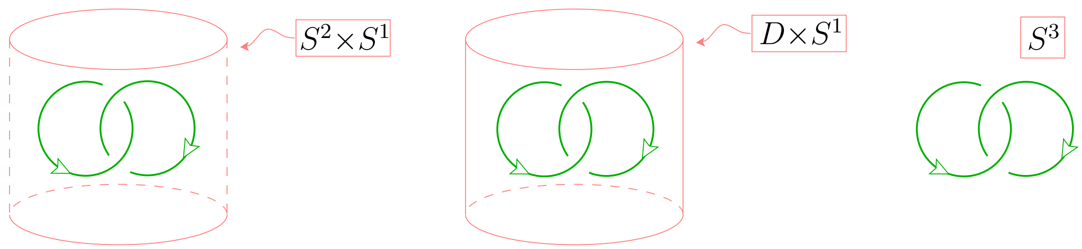
and on the boundary,



(5.12)

(Here the parts enclosed by dashed lines stand for the corresponding parts of figures (5.6) and (5.7).) A slightly lengthy but straightforward calculation, using only the various properties of A and M, i.e. the defining axioms of a symmetric special Frobenius algebra and its representations, shows that this is indeed the case and thus proves independence of the triangulation.

In the remainder of this section we will mainly deal with the following three-manifolds: $S^2 \times S^1$, $D \times S^1$ and S^3, with D denoting a disk. The pictorial representation that we will use for ribbon graphs in these manifolds is illustrated in the following figure, for the example of the Hopf link:



(5.13)

In the first two pictures, the vertical direction corresponds to the S^1 factor, and accordingly top and bottom are to be identified. The first picture stands for the complex number given by the invariant of the relevant ribbon graph (here the Hopf link) in $S^2 \times S^1$; the second picture denotes an element in the vector space $\mathcal{H}(\emptyset, \mathrm{T})$ where T is a two-torus. The third picture again stands for the complex number that is given by the invariant associated to the relevant ribbon graph, but now *multiplied by the factor* $1/S_{0,0}$.

The latter convention avoids a proliferation of factors $S_{0,0}$. Indeed, recall (see formula (2.57) and the subsequent text) that the invariant of a ribbon graph embedded in S^3 is given by $S_{0,0}$ times the complex number obtained from translating the ribbon graph into a morphism in $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$. The convention allows us to replace ribbon graphs in S^3 by morphisms without the need to introduce explicit factors of $S_{0,0}$.

Let us also remark that on orientable world sheets, independence from the triangulation is precisely what is needed for topological invariance in two-dimensional lattice topological theory. Indeed, triangulation independence of our results follows by the same arguments [89] as in two-dimensional lattice TFT. Furthermore, the construction in [89] shows that two-dimensional lattice TFTs are in one-to-one correspondence with symmetric special Frobenius algebras in the category of complex vector spaces. Our construction of conformal field theory amplitudes can therefore be understood [22] as a natural generalisation of lattice TFTs from the category of finite-dimensional complex vector spaces to more general modular tensor categories.

It is also worth mentioning that [118] what is needed in order for triangulations with the same number of faces to yield the same results, are only the properties of A to be a symmetric Frobenius algebra. The property of A to be also special, on the other hand, allows one to reduce the number of faces in a triangulation by the bubble move (5.10). The latter issue can be expected to be much more subtle for irrational conformal field theories.

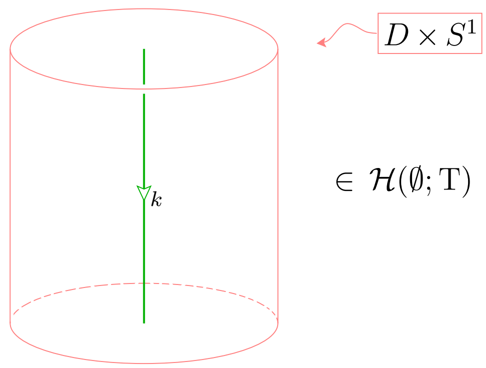
5.2 Zero-point blocks on the torus

The torus and annulus partition functions can both be expressed in terms of conformal blocks on the torus. Let us therefore have a closer look at the space of torus blocks from the TFT point of view.

Let T be the extended surface given by the oriented torus $S^1 \times S^1$, without any embedded arcs (and with a choice of Lagrangian subspace, to be detailed below). As discussed in 2.4, the TFT assigns to T a vector space $\mathcal{H}(\emptyset;\mathrm{T})$ (the symbol $\emptyset$ makes explicit that the extended surface T does not carry any arc). The space $\mathcal{H}(\emptyset;\mathrm{T})$ has dimension $|\mathcal{I}|$. A distinguished basis can be obtained in the following way: Let M_1 be a solid torus with a k-ribbon running along its non-contractible cycle. Choose $\partial_+ M_1 = \mathrm{T}$ and $\partial_- M_1 = \emptyset$, and take the Lagrangian subspace in the first homology of T to be spanned by the cycle in T that is contractible within M_1. The basis vectors $|\chi_k;\mathrm{T}\rangle$ are then obtained by applying the map $Z(M_1,\emptyset,\mathrm{T})$ to the number $1 \in \mathbb{C} = \mathcal{H}(\emptyset)$:

$$|\chi_k;\mathrm{T}\rangle = Z(M_1,\emptyset,\mathrm{T})\,1 \quad \in \mathcal{H}(\emptyset;\mathrm{T})\,. \tag{5.14}$$

In pictures,

$$|\chi_k;\mathrm{T}\rangle \;:=$$  $$\in\; \mathcal{H}(\emptyset;\mathrm{T}) \tag{5.15}$$

Interpreting $\mathcal{H}(\emptyset;\mathrm{T})$ as the space of conformal blocks on the torus, these basis elements correspond to the characters χ_k of the irreducible highest weight modules of the chiral algebra $\mathfrak{V}$. Note that one should actually think of the character as a *one*-point block on the torus, with insertion of the vacuum representation. The character thus depends on the modulus τ as well as on a field insertion $\varphi(v;z)$ with $v \in \mathfrak{V}$ (owing to translation invariance there is no dependence on z). The Virasoro specialised characters are obtained when choosing v to be the vacuum vector, corresponding to $\varphi = \mathbf{1}$, upon which only the τ-dependence is left. When $\mathfrak{V}$ is larger than the VOA associated to the Virasoro algebra, the specialised characters need not be linearly independent. For example, $\chi_k(\tau) = \chi_{\bar{k}}(\tau)$ even when $k \neq \bar{k}$.

Reversing the orientation of T takes the modulus τ to $-\tau^*$. In terms of specialised characters the correspondence is

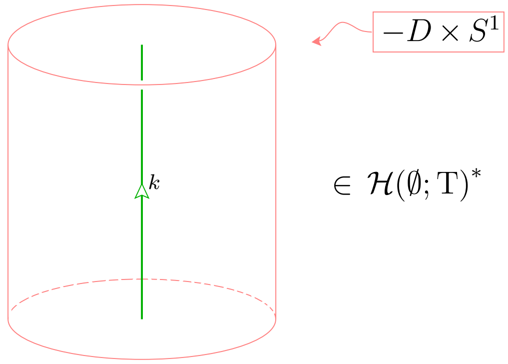
$$|\chi_k;\mathrm{T}\rangle \rightsquigarrow \chi_k(\tau) \qquad \text{and} \qquad |\chi_k;-\mathrm{T}\rangle \rightsquigarrow \chi_k(-\tau^*)\,. \tag{5.16}$$

In order to expand a general element $|\psi;\mathrm{T}\rangle \in \mathcal{H}(\emptyset;\mathrm{T})$ in terms of this basis, we also need the dual basis vectors $\langle\chi_k;\mathrm{T}|$ in $\mathcal{H}(\emptyset;\mathrm{T})^*$. Then we can write $|\psi;\mathrm{T}\rangle = \sum_{k\in\mathcal{I}} \langle\chi_k;\mathrm{T}|\psi;\mathrm{T}\rangle\,|\chi_k;\mathrm{T}\rangle$. Let M_2 be a solid torus with a k-ribbon running along its non-contractible cycle. Take its orientation such that $\partial M_2 = -\mathrm{T}$. We choose $\partial_+ M_2 = \emptyset$ and $\partial_- M_2 = \mathrm{T}$ (recall from section 2.4 that $\partial_- M_2$ is defined to have reversed orientation). Then $Z(M_2,\mathrm{T},\emptyset)$ is a linear function from $\mathcal{H}(\emptyset;\mathrm{T})$ to $\mathbb{C}$, and we have

$$\langle\chi_k;\mathrm{T}| = Z(M_2,\mathrm{T},\emptyset) \quad \in \mathcal{H}(\emptyset;\mathrm{T})^*\,. \tag{5.17}$$

In pictures,

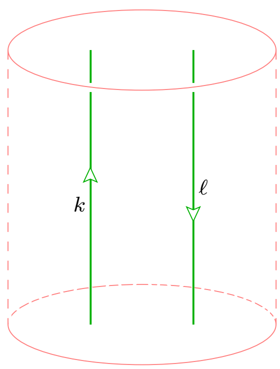
$$\langle \chi_k ; \mathrm{T} | \ =$$



$$\in \ \mathcal{H}(\emptyset;\mathrm{T})^* \tag{5.18}$$

To verify that this is indeed dual to (5.15), use the identity map to glue ∂M_1 to ∂M_2. Then functoriality (2.56) implies $\langle \chi_k;\mathrm{T}\,|\,\chi_\ell;\mathrm{T}\rangle = Z(M_2,\mathrm{T},\emptyset)\circ Z(M_1,\emptyset,\mathrm{T}) = Z(M,\emptyset,\emptyset)$, where M is the three-manifold $S^2\times S^1$, with two ribbons as in the following figure:

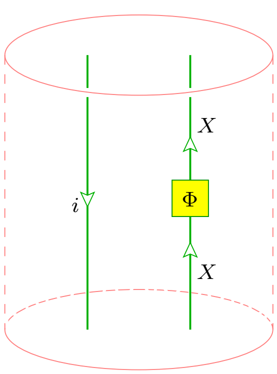
$$\langle \chi_k | \chi_\ell \rangle \ =$$



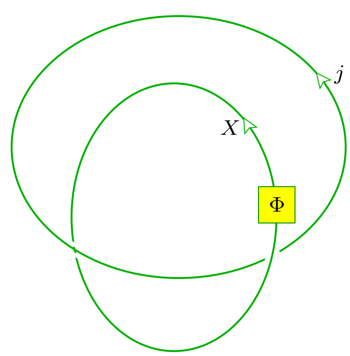
$$= \ \delta_{k,\ell} \tag{5.19}$$

To see how the delta symbol arises, use relation (2.58) to rewrite the number $Z(M,\emptyset,\emptyset)$ as a trace over $\mathcal{H}(k,(\ell,-);S^2)$. Property (2.55) implies that the trace is taken over the identity operator. This leads to $\langle \chi_k;\mathrm{T}\,|\,\chi_\ell;\mathrm{T}\rangle = \dim \mathcal{H}(k,(\ell,-);S^2)$. This dimension, in turn, equals $\delta_{k,\ell}$, as discussed in section 2.4.

By using surgery on the relation (5.19) one can relate the Hopf link (2.22) to the matrix $(S_{i,j})$ of modular transformations of characters. To see this first recall [59] that any link in $S^2\times S^1$ is related to an equivalent link in S^3 via surgery; in the present situation,



$$= \ S_{0,0} \sum_{j\in\mathcal{I}} S_{i,j}$$



$$\tag{5.20}$$

Here X is an arbitrary (not necessarily simple) object and Φ a morphism in $\mathrm{Hom}(X,X)$. Concretely, this result is obtained by cutting out a small solid torus containing the i-ribbon on the left hand side of (5.20) and gluing it back after an S-transformation, yielding the three-manifold S^3 with ribbon graph as on the right hand side.

Upon setting $i = \ell$, $X = U_k$ and $\Phi = id_{U_k}$, (5.20) turns into

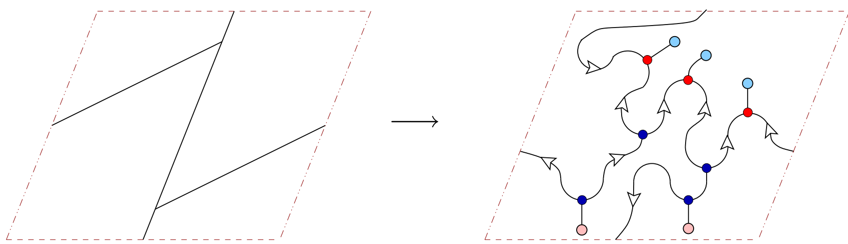
$$\delta_{k,\ell} = S_{0,0} \sum_j S_{\ell,j}\, s_{j,\bar{k}}\,, \tag{5.21}$$

which by $(S^2)_{i,j} = C_{i,j} \equiv \delta_{i,\bar{j}}$ implies that

$$s_{i,j} = S_{i,j}/S_{0,0}\,. \tag{5.22}$$

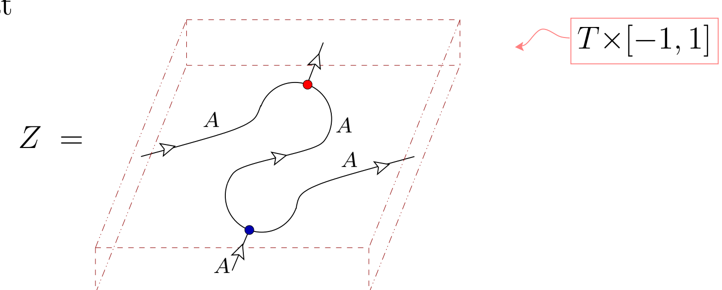
5.3 The torus partition function

Applied to the torus partition function, the general construction in section 5.1 proceeds as follows. The world sheet X is a torus T, thus the double $\hat{\mathrm{X}}$ is the disconnected sum $\mathrm{T} \sqcup (-\mathrm{T})$ of two copies of the torus, and the connecting manifold is $M_{\mathrm{T}} = \mathrm{T} \times [-1, 1]$. Next we pick a triangulation of the world sheet, and then convert this triangulation to a ribbon graph embedded in M_{T}. The left hand side of the following figure displays the triangulation we choose. The ribbon graph in M_T obtained by inserting the elements (5.6) and (5.7) is displayed on the right hand side (the extension in the direction of the interval $[-1, 1]$ is suppressed, all ribbons are labelled by A):



(5.23)

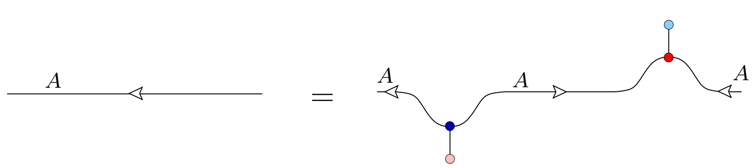
This graph can be simplified a lot using the relations for multiplication and comultiplication of A; this way we arrive at



(5.24)

It is this ribbon graph which we will work with in the rest of this section.

Often we will need a move that reverses the orientation of (the core of) an A-ribbon to see that two given ribbon graphs are equivalent. The following equality on segments of a ribbon graph is a direct consequence of the Frobenius property of A:



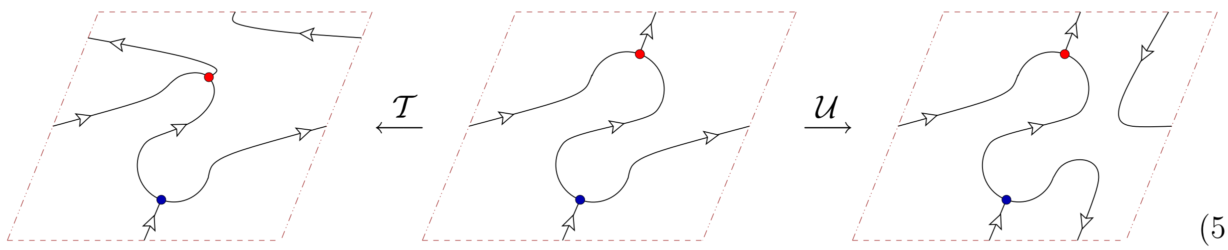
(5.25)

Using this identity, one can reverse the orientation of the core of an A-ribbon. This is achieved as follows: First replace a stretch of A-ribbon with the right hand side of (5.25); then move the multiplication and comultiplication appearing there to the respective ends of the A-ribbon. Whenever both ends of the ribbon end on (the coupon for) a multiplication or comultiplication morphism, one can then use the Frobenius or associativity properties of A to remove the unit and counit.

It was already shown in section 5.1 that the invariant (5.24) is independent of the triangulation we start from. Further, invariance under the action of the relative modular group amounts to the usual modular invariance of the torus partition function. In more detail, consider the two transformations

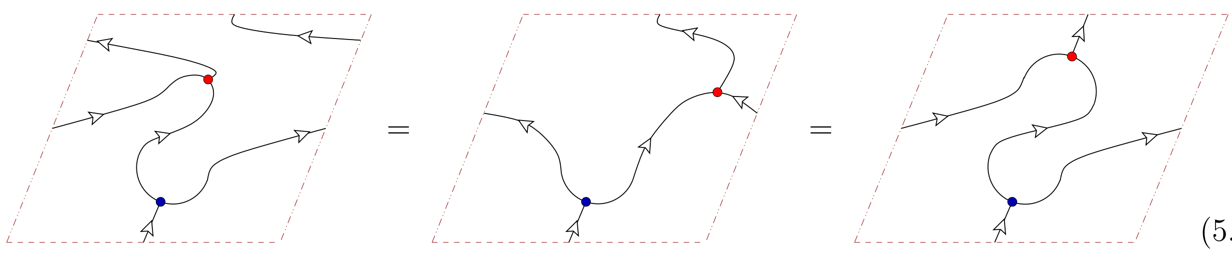
$$\mathcal{U}: \quad \tau \mapsto \tau/(\tau+1) \qquad \text{and} \qquad \mathcal{T}: \quad \tau \mapsto \tau+1 \tag{5.26}$$

of the complex upper half-plane. The map $\mathcal{U}$ corresponds to the change of basis in the lattice that takes $\{\tau, 1\}$ to $\{\tau, 1+\tau\}$. It is expressible in terms of $\mathcal{T}$ and $\mathcal{S}$: $\tau \mapsto -1/\tau$ as $\mathcal{U} = \mathcal{T}\mathcal{S}\mathcal{T}$. These changes of the fundamental region of the torus modify the graph (5.24) as follows:



(5.27)

In agreement with our general arguments about triangulation independence, both resulting graphs can be transformed back to (5.24) by using properties of A. Explicitly, for the change of fundamental domain induced by $\mathcal{T}$ we have:



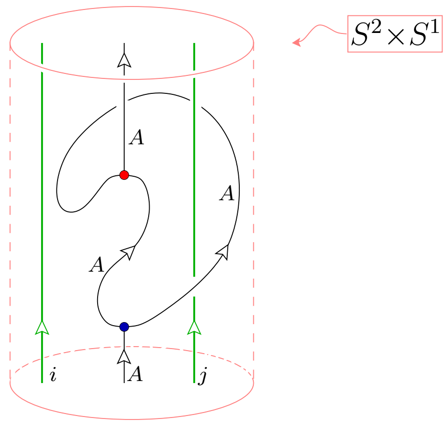
(5.28)

The ribbon graph (5.24) for Z describes an element in $\mathcal{H}(\emptyset; \mathrm{T}) \times \mathcal{H}(\emptyset; -\mathrm{T})$. It can be expanded in a standard basis of conformal blocks (i.e. characters) as

$$Z = \sum_{i,j \in \mathcal{I}} Z_{ij} \, |\chi_i, \mathrm{T}\rangle \otimes |\chi_j, -\mathrm{T}\rangle \,. \tag{5.29}$$

Via the correspondence (5.16), this tells us that in terms of specialised characters of the chiral algebra the CFT partition function reads $Z = \sum_{i,j \in \mathcal{I}} Z_{ij} \, \chi_i(\tau) \, \chi_j(\tau)^*$.

To obtain the coefficients Z_{ij} we glue the dual basis elements for $|\chi_i, \mathrm{T}\rangle$ and $|\chi_j, -\mathrm{T}\rangle$ to the two boundaries of Z. This yields the ribbon graph

Z_{ij} :=  (5.30)

in $S^2 \times S^1$.

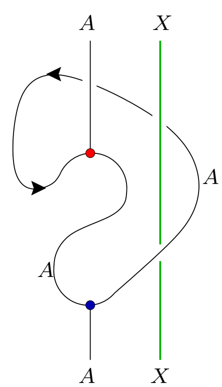
Theorem 5.1:
The numbers Z_{ij} given by the invariant of the ribbon graph (5.30) enjoy the following properties:

$$\text{(i)} \qquad [Z,S]=0 \quad \text{and} \quad [Z,T]=0\,. \tag{5.31}$$

$$\text{(ii)} \qquad Z_{ij} \in \mathbb{Z}_{\geq 0}\,. \tag{5.32}$$

$$\text{(iii)} \qquad Z_{00} = \dim \mathrm{cent}_A(A_{\mathrm{top}})\,. \tag{5.33}$$

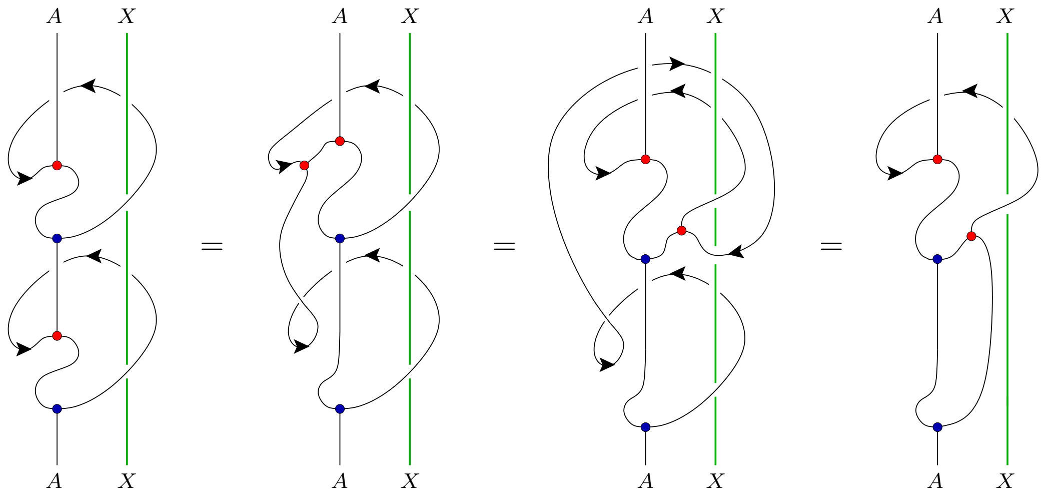
(As usual, we denote the matrices that implement the modular transformations $\mathcal{S}$ and $\mathcal{T}$ on the space of characters by S and T, respectively.) Before proving these claims, it is useful to introduce, for any object X, a specific morphism $P_X \in \mathrm{Hom}(A{\otimes}X, A{\otimes}X)$:

P_X :=  (5.34)

Lemma 5.2:
P_X is a projector:

$$P_X \circ P_X = P_X\,. \tag{5.35}$$

Proof:
It is straightforward to establish the sequence



(5.36)

of equalities for $P_X \circ P_X$. By first using the Frobenius property and then the specialness of A one concludes that the morphism on the right hand side is equal to P_X. ✓

Proof of Proposition 5.1:
Property (i) has already been derived: Z is invariant under T and U, and hence under T and S.
To obtain (ii), cut the picture (5.30) along an S^2 to arrive at a ribbon graph in $S^2 \times [0,1]$, which defines a linear map

$$P_{ij}: \quad \mathcal{H}(i,A,j;S^2) \rightarrow \mathcal{H}(i,A,j;S^2)\,. \tag{5.37}$$

The coefficients Z_{ij} are then recovered as

$$Z_{ij} = \mathrm{tr}_{\mathcal{H}(i,A,j;S^2)}\, P_{ij}\,. \tag{5.38}$$

The morphism described by P_{ij} is nothing but $id_{U_i} \otimes P_j$ with $P_j \equiv P_{U_j}$ as defined in (5.34). Since P_j is a projector, it follows in particular that P_{ij} is a projector as well,

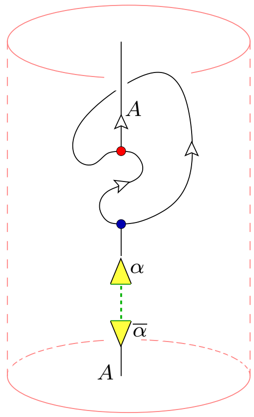
$$P_{ij} \circ P_{ij} = P_{ij}\,. \tag{5.39}$$

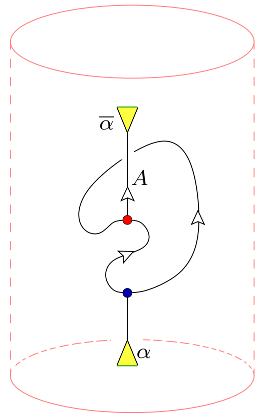
Now the trace of a projector equals the dimension of its image. Hence

$$\mathrm{tr}_{\mathcal{H}(i,A,j;S^2)}\, P_{ij} \in \mathbb{Z}_{\geq 0}\,, \tag{5.40}$$

which establishes (5.32).
To show (iii) we write $Z_{00} = \mathrm{tr}_{\mathcal{H}} P_{00} = \mathrm{tr}_{\mathcal{H}} P_0$ with $\mathcal{H} \equiv \mathcal{H}(A;S^2)$. Then we use dominance to insert a basis in the A-line, leading to $Z_{00} = \sum_{k,\alpha} \mathrm{tr}_{\mathcal{H}}(P_0 \circ \imath^A_{k\alpha} \circ \jmath^{k\bar\alpha}_A)$ (the basis morphisms are

those defined in (3.4)). The space $\mathcal{H}(k;S^2)$ of blocks is non-zero only for $k=0$; thus we obtain

$$\mathrm{tr}_{\mathcal{H}} P_0 \;=\; \sum_\alpha$$


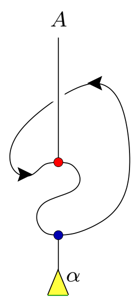
$$=\; \sum_\alpha$$


$$=\; \mathrm{tr}_{A_\mathrm{top}} P_\mathrm{top} \qquad (5.41)$$

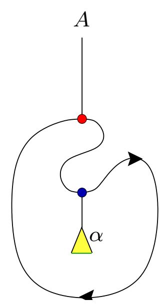
with the linear map

$$\begin{array}{rrcl} P_\mathrm{top}: & A_\mathrm{top} & \to & A_\mathrm{top} \\ & \alpha & \mapsto & P_0 \circ \alpha\,. \end{array} \qquad (5.42)$$

The second equality of (5.41) is just a translation in $S^2 \times S^1$, while in the last step the summation over α is recognised as a trace in A_top. We conclude that $Z_{00} = \dim \mathrm{Im}(P_\mathrm{top})$. Noting the identity



$$=$$



(5.43)

we can now use lemma 3.16 to conclude that $P_\mathrm{top}\alpha = \alpha$ is equivalent to $\alpha \in \mathrm{cent}_A(A_\mathrm{top})$. It follows that $\dim \mathrm{Im}(P_\mathrm{top}) = \dim \mathrm{cent}_A(A_\mathrm{top})$, thus proving formula (5.33). ✓

Let us denote the torus partition function (5.30) by $Z(A)_{ij}$ so as to make its dependence on the algebra object A explicit. Denote further by C the charge conjugation matrix $C_{k\ell} = \delta_{\bar{k},\ell}$ and set

$$\tilde{Z}(A)_{ij} := \sum_{k\in\mathcal{I}} C_{ik}\, Z(A)_{kj}\,. \qquad (5.44)$$

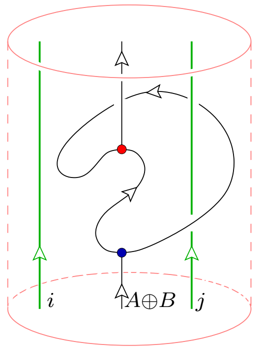
Also recall from section 3.5 the definition of the opposite algebra A_op and of the direct sum $A \oplus B$ and product $A \otimes B$ of algebras.

Proposition 5.3:
The following relations are valid as matrix equations:

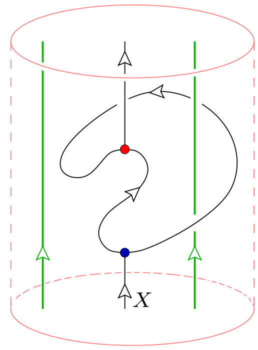
$$\begin{array}{ll} \text{(i)} & \tilde{Z}(A{\oplus}B) = \tilde{Z}(A) + \tilde{Z}(B)\,, \\ \text{(ii)} & \tilde{Z}(A{\otimes}B) = \tilde{Z}(A)\, \tilde{Z}(B)\,, \\ \text{(iii)} & \tilde{Z}(A_\mathrm{op}) = \tilde{Z}(A)^\mathrm{t}\,, \qquad \text{or equivalently,} \qquad Z(A_\mathrm{op}) = Z(A)^\mathrm{t}\,. \end{array} \qquad (5.45)$$

Proof:
(i) The definition of $Z(A)$ gives immediately

$$Z(A{\oplus}B)_{ij} \ =$$

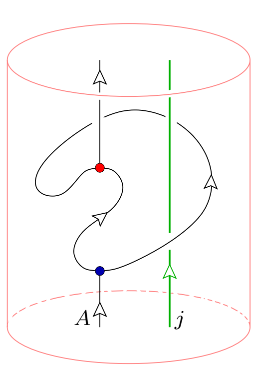


$$= \sum_{X=A,B}$$

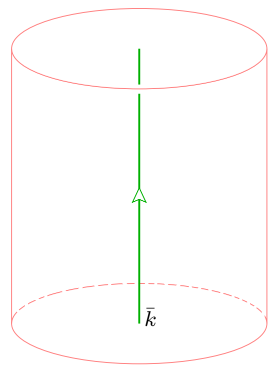


$$= Z(A)_{ij} + Z(B)_{ij} \qquad (5.46)$$

in $S^2 \times S^1$. Multiplying by C from the left yields relation (5.45(i)).
(ii) For ribbon graphs in a solid torus $D \times S^1$ the relation



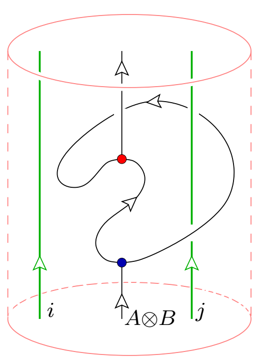
$$= \sum_{k\in\mathcal{I}} Z(A)_{kj}$$



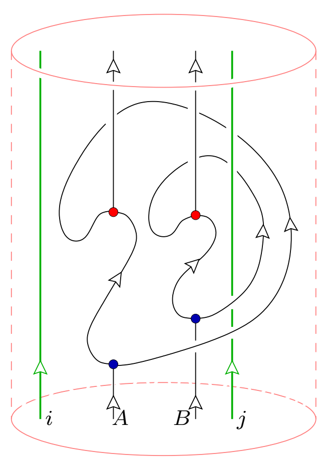
$$(5.47)$$

holds. Here the right hand side is expanded in a basis of zero-point blocks on the torus, as discussed in section 5.2. That the coefficients are precisely given by $Z(A)_{kj}$ can be seen by gluing a $D \times S^1$ containing a single i-ribbon to the left and right hand sides of (5.47). On the left hand side one thereby obtains the graph (5.30) for the torus partition function, while the right hand side gives δ_{ik}. Next note that the ribbon graph for $Z(A{\otimes}B)_{ij}$ can be deformed as

$$Z(A{\otimes}B)_{ij} \ =$$



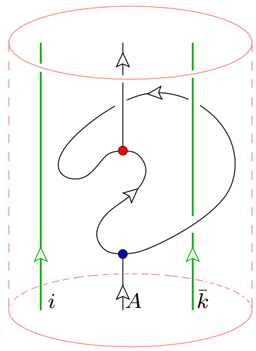
$$=$$



$$(5.48)$$

Now we can apply surgery to cut out a solid torus containing the B- and j-ribbons, and then use (5.47) to obtain

$$Z(A\otimes B)_{ij} \;=\; \sum_{k\in\mathcal{I}} Z(B)_{kj}$$

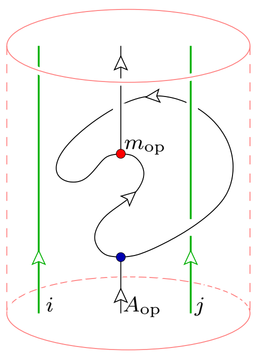


$$=\; \sum_{k\in\mathcal{I}} Z(B)_{kj}\, Z(A)_{i\bar{k}} \;=\; (Z(A)\, C\, Z(B))_{ij}\,. \qquad (5.49)$$

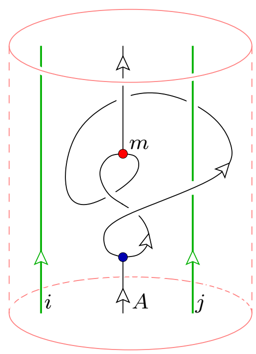
This implies (ii).

(iii) The following relations are valid:

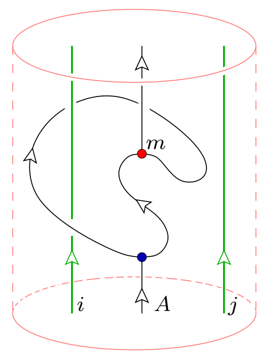
$$Z(A_{\rm op})_{ij} \;=$$



$$=$$



$$=$$



$$=\; Z(A)_{ji}\,. \qquad (5.50)$$

Here in the first step the definition of the multiplication on $A_{\rm op}$ in terms of A is inserted. The second step uses the fact that in the 'horizontal' direction the manifold has the topology of a two-sphere S^2. The A-ribbon is deformed around the two-sphere to wrap around the i-ribbon (this step is best checked by visualising it with actual ribbons, rather than lines, which facilitates keeping track of the twists). Finally we move the j-ribbon to the left side of the graph and use Frobenius and associativity properties to change the A-ribbon so as to match the one appearing in the defining relation (5.30).

Thus we have $Z(A_{\rm op}) = Z(A)^{\rm t}$. This, in turn, is equivalent to $\tilde Z(A_{\rm op}) = C\,\tilde Z(A)^{\rm t}\,C$ and hence (because of $[Z,C]=0$) to $\tilde Z(A_{\rm op}) = \tilde Z(A)^{\rm t}$. ✓

Remark 5.4:

(i) Suppose A and B are algebras such that $Z(A)_{00} = Z(B)_{00} = 1$. Then $Z(A\oplus B)_{00} = 2$. Accordingly the CFT should be interpreted as a superposition of two CFTs with $Z_{00} = 1$, see the brief discussion at the end of section 3.2. Unsurprisingly, it is a superposition of the CFTs associated to A and B, as is confirmed on the level of partition functions by proposition (5.3)(i).

(ii) It is obvious that for two matrices Z_1 and Z_2 that commute with S and T, the product $Z_1 C Z_2$ also commutes with S and T. It is less obvious that this matrix appears as the torus partition function of any consistent CFT. However, when both Z_1 and Z_2 are obtained from algebra objects, then according to the result above $Z_1 C Z_2$ is obtained from an algebra object as well, and is therefore indeed realised as the torus partition function of a CFT. (If the tensor product $A \otimes B$ is isomorphic to a direct sum of algebras, the resulting theory can be interpreted as a superposition of several CFTs with $Z_{00} = 1$, as discussed in point (i).)

(iii) In terms of Morita classes of algebras, the tensor product of CFTs amounts to a product given by $[A] \times [B] := [A \otimes B]$. As an example, consider the WZW theories for $\mathfrak{su}(2)_k$ and $\mathfrak{u}(1)_{2N}$. For $\mathfrak{su}(2)_k$, denote by [A] the Morita class of all algebras giving rise to an A-series modular invariant, and similarly for [D] and for [E6], [E7] and [E8]. The A-invariant occurs at all levels k; its product with itself is $[\mathrm{A}] \times [\mathrm{A}] = [\mathrm{A}]$. In all other cases, too, the multiplication is commutative and [A] acts as unit. Further, one computes

$$\begin{array}{lll} k \in 4\mathbb{Z}: & [\mathrm{D}] \times [\mathrm{D}] = 2\,[\mathrm{D}]\,, & \\ k \in 4\mathbb{Z}{+}2: & [\mathrm{D}] \times [\mathrm{D}] = [\mathrm{A}]\,, & \\ k = 10: & [\mathrm{D}] \times [\mathrm{E}_6] = [\mathrm{E}_6]\,, & [\mathrm{E}_6] \times [\mathrm{E}_6] = 2\,[\mathrm{E}_6]\,, \\ k = 16: & [\mathrm{D}] \times [\mathrm{E}_7] = 2\,[\mathrm{E}_7]\,, & [\mathrm{E}_7] \times [\mathrm{E}_7] = [\mathrm{D}] + [\mathrm{E}_7]\,, \\ k = 28: & [\mathrm{D}] \times [\mathrm{E}_8] = 2\,[\mathrm{E}_8]\,, & [\mathrm{E}_8] \times [\mathrm{E}_8] = 4\,[\mathrm{E}_8]\,. \end{array} \tag{5.51}$$

For $\mathfrak{u}(1)_{2N}$ the situation is more involved. We find

$$[A_{2r}] \times [A_{2s}] = n\,[A_{2t}] \tag{5.52}$$

with integers n and t determined as follows: For $x = r, s$ set $g_x := \gcd(x, N/x)$ and $\alpha_x := N/(x g_x)$. Let a_x be any integer obeying $a_x\,\alpha_x \equiv 1 \bmod (x/g_x)$ (the result does not depend on the particular choice of a_x). Set further $G := \gcd(N/g_r, N/g_s)$. Then n and t are given by

$$n = \gcd(g_r, g_s)\,, \qquad t = \frac{nN}{G\,g_r g_s}\gcd(g_r g_s(a_r\alpha_r - a_s\alpha_s)/n, G)\,. \tag{5.53}$$

One can verify that, as expected, $[A_{2N}]$ is the identity element, and further $[A_{2r}] \times [A_2] = [A_{2N/r}]$ as well as $[A_{2r}] \times [A_{2r}] = \gcd(r, N/r)\,[A_{2\,\mathrm{lcm}(r,N/r)}]$.

Note that the matrices $Z(A_{2r})$ are simple current modular invariants. Products of simple current modular invariants have been considered in [119].

As will be explained in more detail elsewhere, on orientable world sheets bulk fields of the CFT determined by A are triples (i, j, α), where U_i and U_j are simple objects and α is an element of the space $\mathrm{Hom}(A \otimes U_j, U_i^\vee)$ of three-point couplings. However, it turns out that not all triples (i, j, α) are allowed. Instead α has to be *local*, in the sense defined below.

Definition 5.5:
Let A be a symmetric special Frobenius algebra and X, Y objects in $\mathcal{C}$. A morphism φ in the space $\mathrm{Hom}(A \otimes X, Y)$ is called *local* iff $\varphi \circ P_X = \varphi$. Local morphisms in $\mathrm{Hom}(A \otimes X, Y)$ are denoted by

$$\mathrm{Hom_{loc}}(A \otimes X, Y) := \{\varphi \in \mathrm{Hom}(A \otimes X, Y) \,|\, \varphi \circ P_X = \varphi\}\,. \tag{5.54}$$

(The morphism P_X is defined in (5.34).)

Consider a morphism $\alpha \in \mathrm{Hom}(A \otimes U_j, U_{\bar\imath})$. Via $\alpha \mapsto \alpha \circ P_{U_j}$, the morphism P_{U_j} induces a projector p on $\alpha \in \mathrm{Hom}(A \otimes U_j, U_{\bar\imath})$. This follows immediately from (5.36). Denote by $\{\mu_\alpha^{ij}\}$ an eigenbasis of p, i.e.

$$\{\mu_\alpha^{ij}\} \text{ basis of } \mathrm{Hom}(A \otimes U_j, U_{\bar\imath}) \quad \text{s.t.} \quad \mu_\alpha^{ij} \circ P_{U_j} = \varepsilon \mu_\alpha^{ij} \quad \text{with } \varepsilon \in \{0, 1\}\,. \tag{5.55}$$

Denote by $\{\bar{\mu}_\alpha^{ij}\} \subset \mathrm{Hom}(U_{\bar\imath}, A\otimes U_j)$ a basis dual to $\{\mu_\alpha^{ij}\}$, that is (using also dominance)

$$\mu_\alpha^{ij} \circ \bar{\mu}_\beta^{ij} = \delta_{\alpha,\beta}\, id_{\bar\imath} \qquad \text{and} \qquad \sum_{i\in\mathcal{I}} \sum_\alpha \bar{\mu}_\alpha^{ij} \circ \mu_\alpha^{ij} = id_{A\otimes U_j}\,. \tag{5.56}$$

One verifies that $P_{U_j} \circ \bar{\mu}_\alpha^{ij} = \bar{\mu}_\alpha^{ij}$ iff $\mu_\alpha^{ij} \circ P_{U_j} = \mu_\alpha^{ij}$, and that it is zero otherwise.

We also fix the bases $\{\varphi_\alpha^{ij}\}$ of $\mathrm{Hom}(A\otimes U_j, U_i^\vee)$ and $\{\bar\varphi_\alpha^{ij}\}$ of $\mathrm{Hom}(U_i^\vee, A\otimes U_j)$ via

$$\varphi_\alpha^{ij} := \pi_{\bar\imath} \circ \mu_\alpha^{ij} \quad \text{and} \quad \bar\varphi_\alpha^{ij} := \bar\mu_\alpha^{ij} \circ \pi_{\bar\imath}^{-1}\,. \tag{5.57}$$

One quickly checks that the bases $\{\varphi_\alpha^{ij}\}$ and $\{\bar\varphi_\alpha^{ij}\}$ are dual to each other, similar to (5.56), and that

$$\varphi_\alpha^{ij} \text{ local} \quad \Leftrightarrow \quad \mu_\alpha^{ij} \text{ local}\,. \tag{5.58}$$

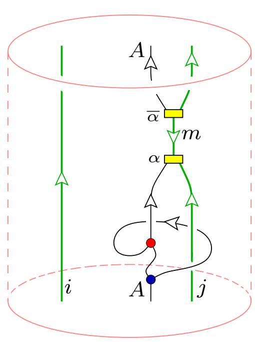
Unless mentioned otherwise, from here on, whenever summing over a basis of morphisms in $\mathrm{Hom}(A\otimes U_i, U_j^\vee)$, $\mathrm{Hom}(A\otimes U_i, U_j)$, $\mathrm{Hom}(U_j^\vee, A\otimes U_i)$ or $\mathrm{Hom}(U_j, A\otimes U_i)$, it is understood that the basis is chosen in the manner described above.

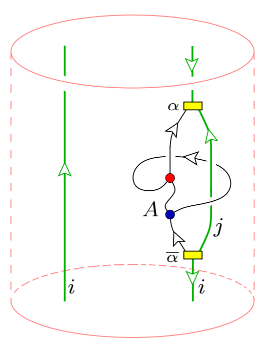
It was stated above that bulk fields are labelled by elements in $\mathrm{Hom}_{\mathrm{loc}}(A\otimes U_j, U_i^\vee)$. This is consistent with the following observation.

Lemma 5.6:
The dimension of the subspace $\mathrm{Hom}_{\mathrm{loc}}(A\otimes U_j, U_i^\vee)$ of local morphisms in $\mathrm{Hom}(A\otimes U_j, U_i^\vee)$ is equal to Z_{ij}:

$$\dim\big[\mathrm{Hom}_{\mathrm{loc}}(A\otimes U_j, U_i^\vee)\big] = Z_{ij}\,. \tag{5.59}$$

Proof:
Using dominance and the fact that $\mathcal{H}(i,(m,-);S^2)$ is non-zero only for $i=m$, we can rewrite Z_{ij} from (5.30) as

$Z_{ij} \;=\; \sum_{m,\alpha}$ 

$= \sum_{\alpha}$ 

(5.60)

Here the sum over α runs over the basis introduced in (5.57). The last ribbon graph in (5.60) can be seen to contain the element $\varphi_\alpha^{ij} \circ P_{U_j}$, thus the sum can be restricted to a basis of $\mathrm{Hom}_{\mathrm{loc}}(A\otimes U_j, U_i^\vee)$. By definition of the basis φ_α^{ij} we can replace $\varphi_\alpha^{ij} \circ P_{U_j}$ by φ_α^{ij} in the last graph of (5.60). But now φ_α^{ij} and its dual cancel to $id_{U_i^\vee}$ and the resulting ribbon graph takes the constant value 1. The sum is thus equal to the number of local basis elements in $\mathrm{Hom}(A\otimes U_j, U_i^\vee)$. ✓

5.4 Bulk fields and α-induced bimodules

We now further investigate the space of bulk fields. This allow us in particular to show that our prescription for the modular invariant torus partition function coincides with the one obtained by different methods in [32], and at the same time give a deeper understanding of the space $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X,Y)$ of local morphisms. Bimodules over A are a crucial ingredient in this analysis, so we start with a few comments on the category ${}_A\mathcal{C}_A$ of A-A-bimodules. In contrast to the category of left A-modules, ${}_A\mathcal{C}_A$ is naturally endowed with the structure of a tensor category, with tensor product $M \otimes_A N$ defined to be the tensor product over A. The tensor unit of ${}_A\mathcal{C}_A$ is A itself. We denote the morphism spaces of ${}_A\mathcal{C}_A$ by $\mathrm{Hom}_{A|A}(\cdot\,,\cdot)$.

The following prescription defines two tensor functors $\alpha^{(\pm)}$ from $\mathcal{C}$ to ${}_A\mathcal{C}_A$. For every object V of $\mathcal{C}$ the induced left module $\mathrm{Ind}_A(V) = (A{\otimes}V, m{\otimes}id_V)$ can be endowed with two different structures of a right A-module: In the first case the right action $\rho_\mathrm{r} \in \mathrm{Hom}(A \otimes V \otimes A, A)$ is

$$\rho_\mathrm{r}^{(+)} := (m \otimes id_V) \circ (id_A \otimes c_{V,A})\,, \tag{5.61}$$

and in the second case,

$$\rho_\mathrm{r}^{(-)} := (m \otimes id_V) \circ (id_A \otimes (c_{A,V})^{-1})\,. \tag{5.62}$$

Since A is associative, both of these right actions commute with the left action of A on $A \otimes V$. We denote the two induced bimodules obtained this way by $\alpha^{(\pm)}(V)$. (In the introduction we have used the notation $(A{\otimes}V)^\pm$ instead, because it adapts easier to the general situation, where one has $(B{\otimes}V)^\pm$ with V an object of $\mathcal{C}$ and B an arbitrary A-A-bimodule.) Both functors $\alpha^{(\pm)}$ act on morphisms $f \in \mathrm{Hom}(V,W)$ in the same way as the induction functor of left modules:

$$\alpha^{(\pm)}(f) := id_A \otimes f \;\in \mathrm{Hom}(A \otimes V, A \otimes W)\,. \tag{5.63}$$

The two tensor functors $\alpha^{(\pm)}$ have first appeared in the theory of subfactors [120], where they play a crucial role (see e.g. [121,122,123,124,125]) and have been termed *α-induction*. The category-theoretic reformulation presented here was obtained in [13]. For those cases where A describes the embedding of a subfactor into a factor, it has been shown [32] that the matrix with entries

$$Z^{(\mathrm{BEK})}_{k,l} := \dim\big[\mathrm{Hom}_{A|A}(\alpha^{(-)}(U_k), \alpha^{(+)}(U_l))\big] \tag{5.64}$$

$(k,l \in \mathcal{I})$ commutes with the matrices that describe the action of the modular group and hence provides a modular invariant.

We now show that the matrix (5.64) is related to the modular invariant partition function discussed in section 5.3 above by

$$Z^{(\mathrm{BEK})}_{k,l} = Z_{\bar l,k}\,. \tag{5.65}$$

Along with the construction of general correlators outlined in [22], this shows in particular that $Z^{(\mathrm{BEK})}$ is realised as the torus partition function of a physical conformal field theory, a property that does not follow from the subfactor considerations.

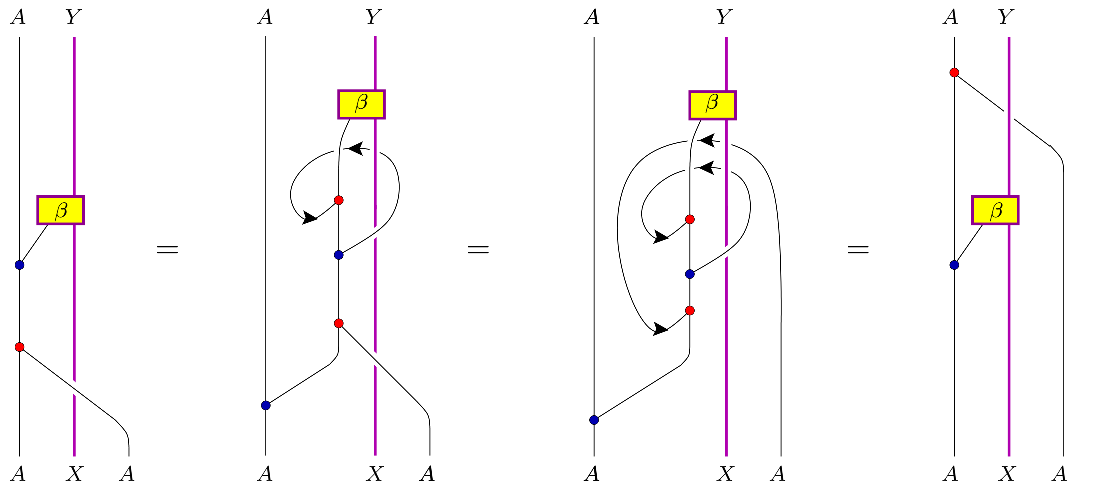
To derive the relation (5.65) we make use of lemma 5.6, in view of which we need to relate the space $\mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X,Y)$ of local morphisms to morphisms of α-induced bimodules. This is done as follows.

Proposition 5.7:
For any symmetric special Frobenius algebra A in a braided tensor category the linear map

$$\begin{aligned} f:\quad \mathrm{Hom}_{\mathrm{loc}}(A{\otimes}X,Y) &\longrightarrow \mathrm{Hom}_{A|A}(\alpha^-(X),\alpha^+(Y)) \\ \beta &\longmapsto (id_A{\otimes}\beta)\circ(\Delta{\otimes}id_X) \end{aligned} \tag{5.66}$$

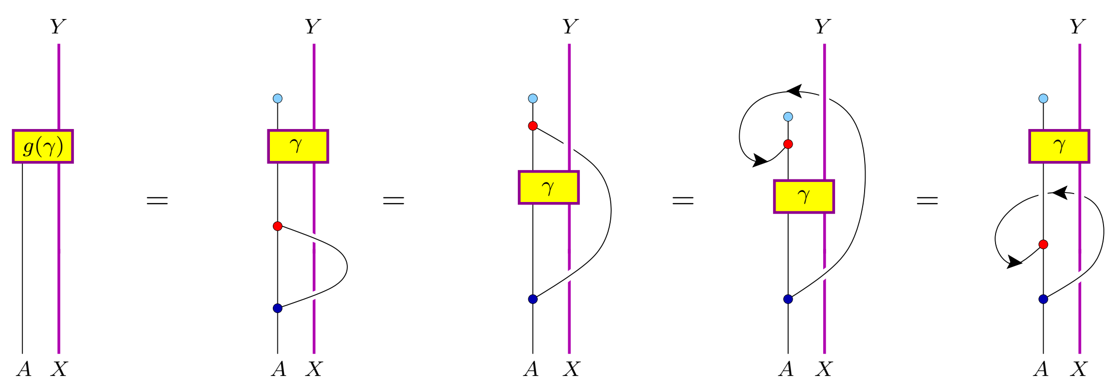
is an isomorphism.
Its (left and right) inverse is given by g: $\gamma \mapsto (\varepsilon \otimes id_Y) \circ \gamma$ for $\gamma \in \mathrm{Hom}_{A|A}(\alpha^-(X), \alpha^+(Y))$.

Proof:
We first show that f maps into the correct space. That f is a linear map from $\mathrm{Hom}_{\mathrm{loc}}(A \otimes X, Y)$ to $\mathrm{Hom}(\alpha^-(X), \alpha^+(Y))$ is obvious. What we still must check is that $f(\beta)$ is in fact even in $\mathrm{Hom}_{A|A}$, i.e. a morphism of A-A-bimodules; this is done as follows.
It is a direct consequence of the Frobenius property of A that for every $\beta \in \mathrm{Hom}_{\mathrm{loc}}(A \otimes X, Y)$ the image $f(\beta)$ is a morphism of left A-modules. To see that it is a morphism of right A-modules as well, we consider the equalities



(5.67)

In the first equality one uses the Frobenius property of A and the locality property of β to insert a projector P_X (as defined in formula (5.34)). The second step summarises a number of moves that take the A-ribbon around the projector; these employ the Frobenius and associativity properties of A. The third step uses again that β is local, which allows one to leave out the projector, and that A is symmetric Frobenius to take the A-ribbon past the comultiplication. Notice that on $A \otimes X$ the action of A from the right is defined via the braiding, while on $A \otimes Y$ it is defined using the inverse of the braiding.
Next we establish that g maps into the correct space, too. By definition, g is a linear map from $\mathrm{Hom}_{A|A}(\alpha^-(X), \alpha^+(Y))$ to $\mathrm{Hom}(A \otimes X, Y)$. It remains to show that g is local.
For every $\gamma \in \mathrm{Hom}_{A|A}(\alpha^-(X), \alpha^+(Y))$ we have the equalities



(5.68)

In the first step the definition of $g(\gamma)$ is inserted, and it is used that A is special. The second step follows because γ is a morphism of right A-modules. The third step employs symmetry of

A, and the last one that γ is a left A-morphism. Comparing the left and right sides of (5.68), we see that $g(\gamma) = g(\gamma) \circ P_X$, which means that indeed $g(\gamma) \in \mathrm{Hom}_{\mathrm{loc}}(A \otimes X, Y)$.
Finally we show that f and g are each others' inverses. That $g \circ f = id$ on $\mathrm{Hom}_{\mathrm{loc}}(A \otimes X, Y)$ is an immediate consequence of the defining property of the counit. To see that also $f(g(\gamma)) = \gamma$ for all $\gamma \in \mathrm{Hom}_{A|A}(\alpha^-(X), \alpha^+(Y))$ one uses that A is Frobenius and that γ is in particular a left A-morphism. These properties imply that f is injective and surjective. ✓

5.5 The extended left and right chiral algebras

Let us recall that the chiral algebra $\mathfrak{V}$, which corresponds to a subsector of the holomorphic fields of a CFT, is the vertex operator algebra whose representation theory gives rise to the modular category $\mathcal{C}$. The full, maximally extended, chiral algebra of the CFT can be larger than $\mathfrak{V}$. It can also be different for the left and right (holomorphic and anti-holomorphic) parts of the CFT. Let us denote these maximally extended left and right chiral algebras by A_ℓ and A_r, respectively. Their $\mathfrak{V}$-representation content can be read off the torus partition function Z as

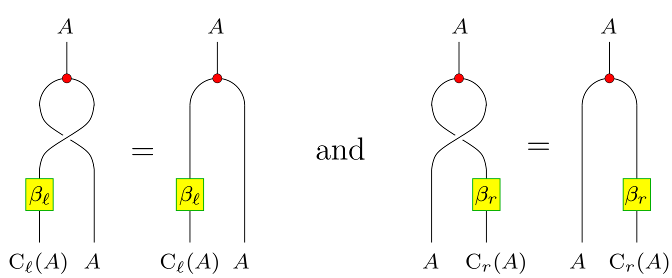
$$A_\ell \cong \bigoplus_{j \in \mathcal{I}} Z_{j0}\, U_j \qquad \text{and} \qquad A_r \cong \bigoplus_{i \in \mathcal{I}} Z_{0i}\, U_i \,. \tag{5.69}$$

These left and right chiral algebras also turn out to possess a nice interpretation in terms of the algebra object A. To this end let us introduce the notion of the left and right center of an algebra [13].

Definition 5.8:
The *left center* $\mathrm{C}_\ell(A)$ of an algebra A is the maximal subobject of A such that

$$m \circ c_{A,A} \circ (\beta_\ell \otimes id_A) = m \circ (\beta_\ell \otimes id_A) \,, \tag{5.70}$$

with $\beta_\ell \in \mathrm{Hom}(\mathrm{C}_\ell(A), A)$ is the embedding morphism of $\mathrm{C}_\ell(A)$ into A.
The *right center* $\mathrm{C}_r(A)$ of an algebra A is the maximal subobject of A such that the equality $m \circ c_{A,A} \circ (id_A \otimes \beta_r) = m \circ (id_A \otimes \beta_r)$ holds.
In pictures,



(5.71)

As already mentioned, two-dimensional conformal field theories that are even topological are included in our construction by specialising to the tensor category of complex vector spaces. In this specific context, the idea has been put forward [126,127,128] that the boundary fields yield a non-commutative Frobenius algebra – the algebra of open string states – and that its center describes an algebra of bulk fields (closed string states). We will now see how the situation looks like for general conformal field theories.

An important property [13] of the left and right centers of an algebra A is that they are invariant under Morita equivalence. This is consistent with the fact that the CFT only depends on the Morita class of the algebra A, so that physical quantities extracted from the algebra A must be invariant under Morita equivalence, too. Furthermore, the left and right chiral algebras are precisely given by the left and right centers of A (see claim 5 in [13]), i.e.:

Proposition 5.9:
As objects in $\mathcal{C}$,

$$A_\ell \cong \mathrm{C}_\ell(A) \qquad \text{and} \qquad A_r \cong \mathrm{C}_r(A)\,. \tag{5.72}$$

The proof of proposition 5.9 will fill the remainder of this section. Let us start by giving an alternative definition of the left and right centers:

Definition 5.10:
The *left center* $\mathrm{cent}_{A,\mathrm{left}}$ of an algebra A is the operation that assigns to any object X of $\mathcal{C}$ the subspace of all elements in $\mathrm{Hom}(X,A)$ such that the multiplication becomes commutative (with respect to the braiding $c_{A,A}$), i.e.

$$\mathrm{cent}_{A,\mathrm{left}}(X) := \{\, \alpha \in \mathrm{Hom}(X,A) \mid m \circ c_{A,A} \circ (\alpha \otimes id_A) = m \circ (\alpha \otimes id_A) \,\}\,. \tag{5.73}$$

Analogously, the *right center* $\mathrm{cent}_{A,\mathrm{right}}$ of an algebra A is defined by

$$\mathrm{cent}_{A,\mathrm{right}}(X) := \{\, \alpha \in \mathrm{Hom}(X,A) \mid m \circ c_{A,A} \circ (id_A \otimes \alpha) = m \circ (id_A \otimes \alpha) \,\}\,. \tag{5.74}$$

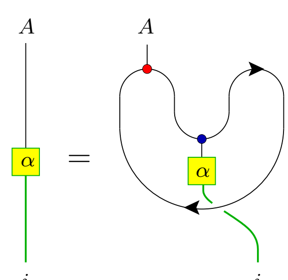
The relation to the previous definition is

$$\mathrm{C}_\ell(A) \cong \bigoplus_{i\in\mathcal{I}} \dim\left[\mathrm{cent}_{A,\mathrm{left}}(U_i)\right] U_i\,, \tag{5.75}$$

and similarly for $\mathrm{C}_r(A)$. Thus in this alternative formulation the relevant subobject of A is characterised via the components of its embedding morphism. There is also a close relationship with the relative center of A_{top} as introduced in definition 3.15:

$$\mathrm{cent}_A(A_{\mathrm{top}}) = \mathrm{cent}_{A,\mathrm{left}}(\mathbf{1}) = \mathrm{cent}_{A,\mathrm{right}}(\mathbf{1})\,. \tag{5.76}$$

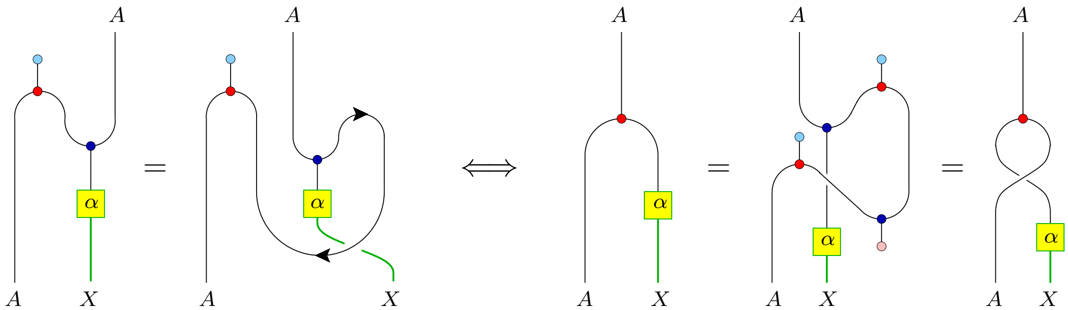
Lemma 5.11:
The morphism $\alpha \in \mathrm{Hom}(U_i, A)$ is in $\mathrm{cent}_{A,\mathrm{right}}(U_i)$ iff



(5.77)

An analogous statement holds for the left center when the inverse braiding is used in (5.77).

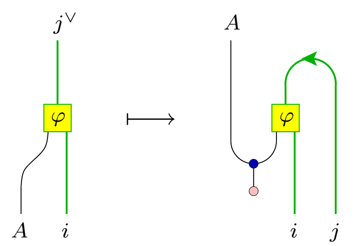
Proof:
To see the relation for the right center, consider the equivalence



(5.78)

The first of these equalities is equivalent to the left equality in (3.61); indeed, it can be obtained by composing $id \otimes (3.61)$ with $(\varepsilon \circ m) \otimes id_A$. Further, lemma 3.17 implies that this first equality is, in turn, equivalent to the relation (5.77). To get the equivalence asserted in (5.78), one uses the Frobenius property several times, on both sides of the first equality, as well as the fact that A is symmetric, i.e. relation (3.48).
For the left center the proof proceeds analogously. ✓

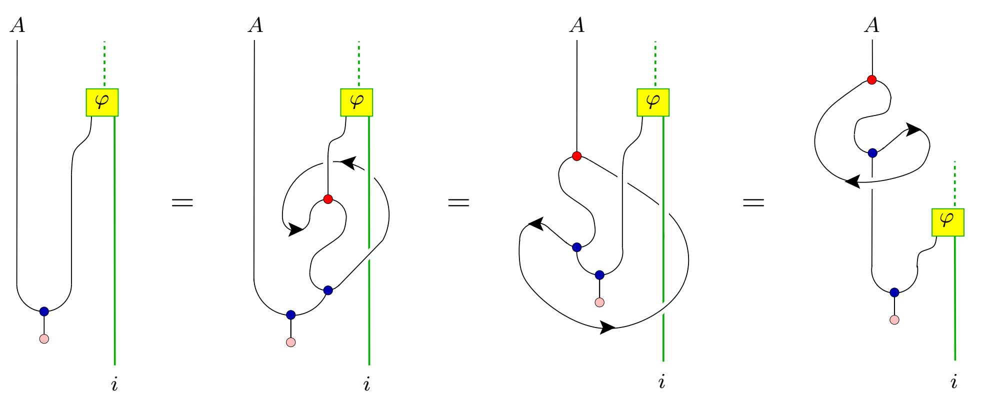
To proceed, we introduce, for all $i, j \in \mathcal{I}$, the isomorphism $L_{ij}\colon \mathrm{Hom}(A \otimes U_i, U_j^\vee) \to \mathrm{Hom}(U_i \otimes U_j, A)$ via



(5.79)

Lemma 5.12:
For each $i \in \mathcal{I}$, the map L_{i0} defines an isomorphism between $\mathrm{cent}_{A,\mathrm{right}}(U_i)$ and $\mathrm{Hom}_{\mathrm{loc}}(A \otimes U_i, \mathbf{1})$.

Proof:
Consider the moves

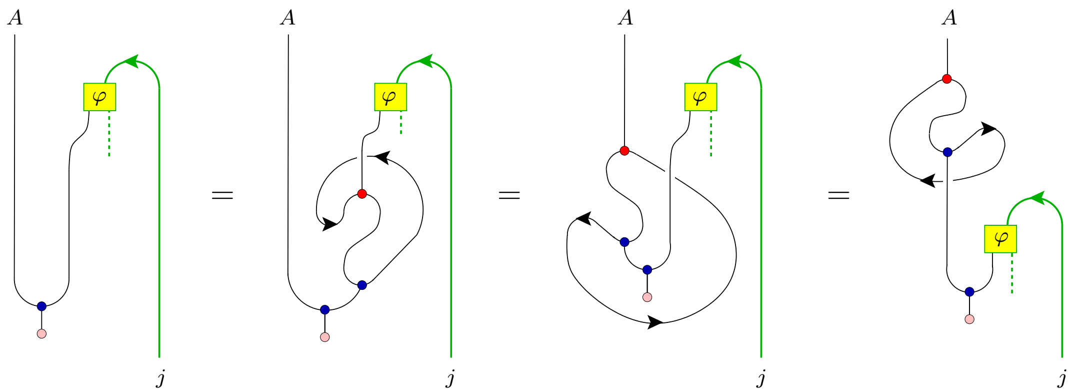


(5.80)

The first equality is the definition of φ being local. In the equivalence we composed both sides with $(\Delta \circ \varepsilon) \otimes id_{U_i}$. The right hand side of the resulting equation can be rewritten as sketched above by using that A is symmetric and Frobenius. The resulting equality is equivalent, by lemma 5.11 to $L_{i0}(\varphi)$ being in the right center. ✓

Lemma 5.13:
For each $j \in \mathcal{I}$, the map L_{0j} defines an isomorphism between $\mathrm{cent}_{A,\mathrm{left}}(U_j, A)$ and the subspace of local elements in $\mathrm{Hom}(A \otimes \mathbf{1}, U_j^\vee)$.

Proof:
This is shown similarly to the previous lemma. The manipulations look as follows:



(5.81)

The left equality is again the definition of φ being local, while by lemma 5.11 the right equality is equivalent to φ being in the left center. ✓

Proof of proposition 5.9:
According to lemma 5.6, the dimension of the subspace of local elements in $\mathrm{Hom}(A \otimes U_j, U_i^\vee)$ equals Z_{ij}. Thus by the lemmata 5.12 and 5.13 it follows that

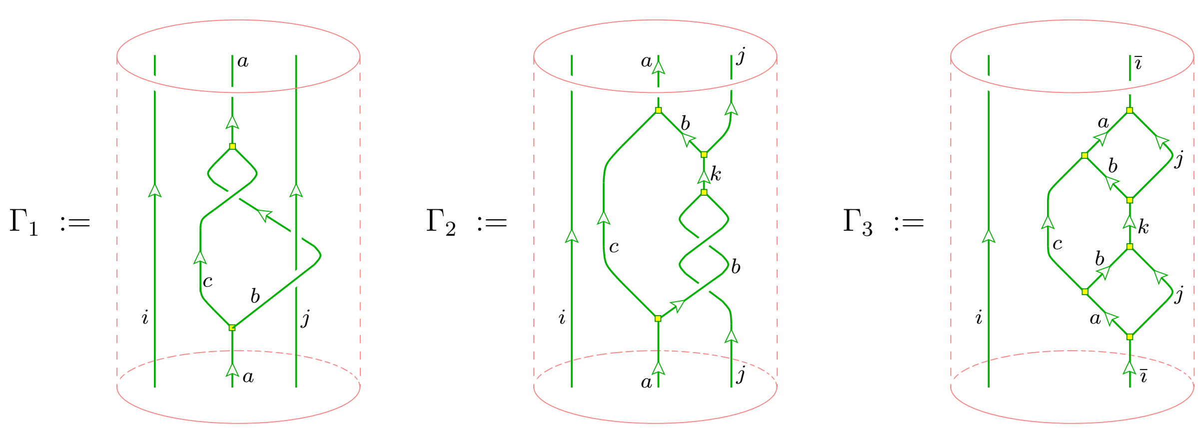
$$Z_{0i} = \dim\big[\mathrm{cent}_{A,\mathrm{right}}(U_i)\big] \qquad \text{and} \qquad Z_{j0} = \dim\big[\mathrm{cent}_{A,\mathrm{left}}(U_j)\big]\,. \tag{5.82}$$

As a consequence, the definition of A_ℓ and A_r in equation (5.69) does coincide with the definition 5.8 of $\mathrm{C}_\ell(A)$ and $\mathrm{C}_r(A)$. ✓

5.6 The case ${N_{ij}}^k \in \{0,1\}$ and $\dim \mathrm{Hom}(U_k, A) \in \{0,1\}$

In this section we express the ribbon graph (5.30) giving the torus partition function in terms of the Moore–Seiberg data of the modular category $\mathcal{C}$. For notational simplicity we concentrate on the situation that both ${N_{ij}}^k \in \{0,1\}$ for all $i,j,k \in \mathcal{I}$ and $\dim \mathrm{Hom}(U_k, A) \in \{0,1\}$ for all $k \in \mathcal{I}$.

The ribbon graphs defined by



(5.83)

can be related to each other by inserting bases as described in sections 3.1 and 3.6, using the compatibility constraint between braiding and twist, and accounting for $\dim \mathcal{H}(i,j;S^2)=\delta_{j,\bar\imath}$. This way we simplify the expression for the ribbon graph (5.30) to

$$\begin{aligned} Z_{ij} &= \sum_{a,b,c\prec A} {m_{bc}}^{a} \Delta_a^{cb}\, \theta_b\, \Gamma_1 = \sum_{a,b,c\prec A} \sum_{k\in\mathcal{I}} {m_{bc}}^{a} \Delta_a^{cb}\, \theta_b\, \mathsf{R}^{(c\,b)a}\, \Gamma_2 \\ &= \sum_{a,b,c\prec A} \sum_{k\in\mathcal{I}} {m_{bc}}^{a} \Delta_a^{cb}\, \frac{\theta_k}{\theta_j}\, \mathsf{R}^{(c\,b)a}\, \Gamma_3\,. \end{aligned} \tag{5.84}$$

Via a fusion and an inverse fusion operation one then arrives at a graph that can be further simplified by using the definition of the dual coupling. The final result is

$$Z_{ij} = \sum_{a,b,c\prec A} {m_{bc}}^{a} \Delta_a^{cb} \sum_{k\in\mathcal{I}} \mathsf{G}^{(c\,b\,j)\,\bar\imath}_{a\,k}\, \mathsf{R}^{(c\,b)a}\, \frac{\theta_k}{\theta_j}\, \mathsf{F}^{(c\,b\,j)\,\bar\imath}_{k\,a}\,. \tag{5.85}$$

(Recall that the coproduct can also be expressed in terms of the product using (3.83).) Note that in this formulation, it is not obvious why Z_{ij} should be real, let alone a non-negative integer. This illustrates the power of the graphical approach, which readily supplies a proof of non-negativity and integrality.

5.6.1 Example: Free boson

The expression (5.85) can now be used to compute the torus partition function associated to the algebra A_{2r} as defined in (3.84) and (3.85). We also need the $\mathbb{Z}_{2N}$ free boson modular data given in (2.77). Substituting these into formula (5.85), one notes that the simple fusion rules make three of the four sums disappear, and we end up with

$$Z_{[x][y]}(A_{2r}) = \delta_{[x+y]\prec A} \cdot \frac{r}{N} \sum_{a\prec A} \exp(2\pi\mathrm{i}\, \tfrac{x-y}{2}\, \tfrac{a}{2N}) = \delta_{x+y,0 \bmod 2r}\, \delta_{x-y,0 \bmod 2N/r}\,. \tag{5.86}$$

Let us have a look at the two extremal cases. For $r=N$ we have $A_{2r}=A_{2N}=[0]$. This leads to

$$Z_{[x][y]}(A_{2N}) = \delta_{x+y,2N}\,. \tag{5.87}$$

(Recall that we chose representatives of simple objects such that $0\le x,y<2N$.) The partition function(5.87) is just the charge conjugation modular invariant. On the other extreme, for $r=1$ we deal with $A_2=[0]\oplus[2]\oplus\cdots\oplus[2N{-}2]$ and find

$$Z_{[x][y]}(A_2) = \delta_{x,y}\,, \tag{5.88}$$

which pairs $\mathfrak{V}$-representations of the same (minimal) $\mathfrak{u}(1)$ charge; this is precisely the T-dual of the partition function (5.87).

More generally, one easily verifies that

$$Z_{[x][2N-y]}(A_{2r}) = Z_{[x][y]}(A_{2N/r})\,, \tag{5.89}$$

which means that the two algebras A_{2r} and $A_{2N/r}$ produce T-dual theories.

5.6.2 Example: E_7 modular invariant

It is only now that we can verify that the title of this example indeed deserves its name. We combine the definition of the modular category for $\mathfrak{su}(2)_k$ in section 2.5.2 at level $k=16$, the explicit form (3.90) for the product of A, and the formula (5.85) for the coefficients Z_{ij} in $Z=\sum_{i,j\in\mathcal{I}} Z_{ij}\,\chi_i\,\chi_j^*$. We can then check numerically that we get precisely the E_7 modular invariant:

$$Z = |\chi_0{+}\chi_{16}|^2 + |\chi_4{+}\chi_{12}|^2 + |\chi_6{+}\chi_{10}|^2 + |\chi_8|^2 + \chi_8\,(\chi_2^*+\chi_{14}^*) + (\chi_2+\chi_{14})\,\chi_8^*\,. \tag{5.90}$$

Incidentally, one also checks that A of the form

$$A = (0)\oplus(k/2)\oplus(k)\,, \tag{5.91}$$

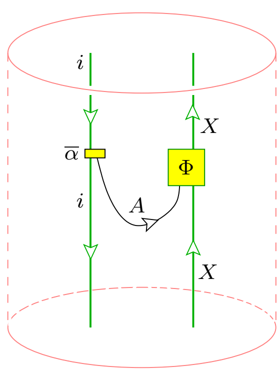
i.e. precisely containing the unit, the simple current and its fixed point (see (3.90)), is an algebra also at the levels $k=4$ and $k=8$. In the first case it is Morita equivalent to $\mathbf{1}$, since (recall from section 3.6.2 that the multiplication on $\mathbf{1}{\oplus}f{\oplus}J$ is unique)

$$(0)\oplus(2)\oplus(4)\cong(2)\otimes(2)\,, \tag{5.92}$$

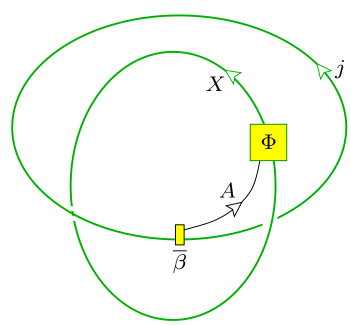
and accordingly gives rise to the A-series modular invariant, while in the second case one obtains the D-series modular invariant.

5.7 One-point blocks on the torus

Similarly as for the zero-point blocks on the torus, we may study the behavior of torus blocks with one A-insertion, i.e. of elements of $\mathcal{H}(A;\mathrm{T})$, under an $\mathcal{S}$-transformation. In analogy with (5.20) we fix our conventions by defining the matrix $L^A=(L^A_{i\alpha,j\beta})$ via



$$= S_{0,0}\sum_{j,\beta} L^A_{i\alpha,j\beta} \tag{5.93}$$

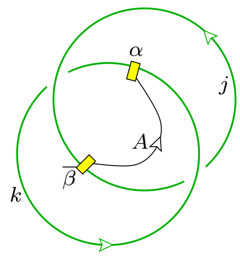


Here X is an arbitrary (not necessarily simple) object, $\Phi\in\mathrm{Hom}(A{\otimes}X,X)$, U_i,U_j are simple objects, $\bar\alpha$ denotes a basis vector in $\mathrm{Hom}(U_i,A{\otimes}U_i)$, and $\bar\beta$ runs over the basis in $\mathrm{Hom}(U_j,A{\otimes}U_j)$. That the ribbon graphs on the two sides are indeed linearly related – and thus that the matrix L^A exists – follows from the general result that the TFT provides us with a projective representation of the mapping class group.

Let us examine some properties of L^A.

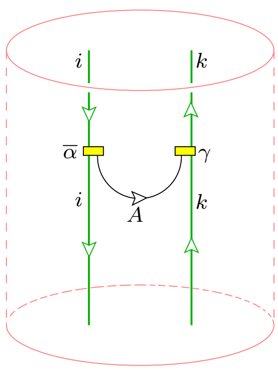
Lemma 5.14:
The matrix L^A is invertible, with inverse $\tilde{L}^A$ given by

$$\tilde{L}^A_{j\beta,k\gamma} \;=\; S_{0,0} \tag{5.94}$$

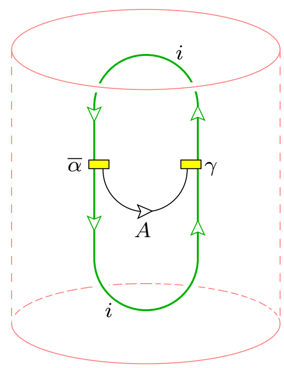


Proof:
For $X = U_k$ and $\Phi = \gamma$, the left hand side of (5.93) reads

l.hs. $=$



$= \dfrac{\delta_{ik}}{\dim(U_i)}$



$= \delta_{i,k}\,\delta_{\alpha,\gamma}\,,$ (5.95)

while the ribbon graph on the right hand side of (5.93) reduces to $\tilde{L}^A_{j\beta,k\gamma}/S_{0,0}$, so that (5.93) becomes

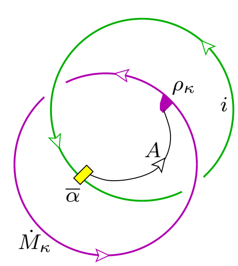
$$\delta_{i,k}\,\delta_{\alpha,\gamma} = \sum_{j\in\mathcal{I}}\sum_{\beta} L^A_{i\alpha,j\beta}\,\tilde{L}^A_{j\beta,k\gamma}\,. \tag{5.96}$$

Thus L^A has $\tilde{L}^A$ as its right-inverse. Since L^A is a square matrix, this is also a left-inverse. ✓

Note that the relation (5.21) for the zero-point blocks is recovered as a special case of (5.96), by setting $A = \mathbf{1}$ and $\alpha = \gamma = id$. In this case $\tilde{L}^A$ is just the inverse of the S-matrix, i.e. $\tilde{L}^{\mathbf{1}}_{j,k} = S_{0,0}\, s_{j,\bar{k}}$ with s as defined in (2.22).

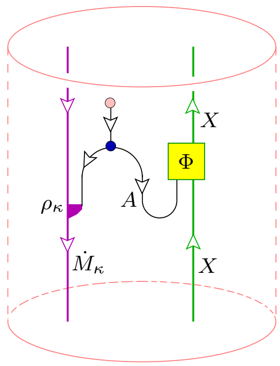
In the next section we will heavily use matrices $S^A = (S^A_{\kappa,i\alpha})$ and $\tilde{S}^A = (\tilde{S}^A_{i\alpha,\kappa})$, where $\kappa\in\mathcal{J}$ (corresponding to a simple A-module M_κ) and $i\in\mathcal{I}$ (corresponding to a simple object U_i), and $\alpha\in\mathrm{Hom}(A{\otimes}U_i, U_i)$ is local. We start by defining

$$\tilde{S}^A_{i\alpha,\kappa} \;:=\; S_{0,0} \tag{5.97}$$

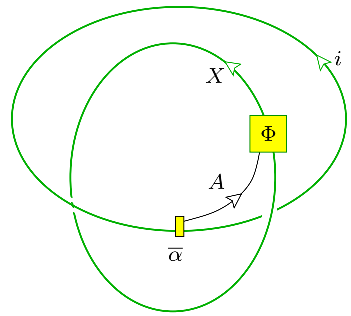


Note that the row and column labels of $\tilde{S}^A$ take their values in two rather different index sets. (Thus, while being similar to the S-matrix for the category of local A-modules that is used in [87], it is not quite the same matrix. The notion of a local module as defined in [87] refers to algebras A that are commutative and have trivial twist, neither of which properties is imposed here.)

The matrix S^A is defined to implement an $\mathcal{S}$-transformation on the torus one-point blocks, analogously to L^A in (5.93):



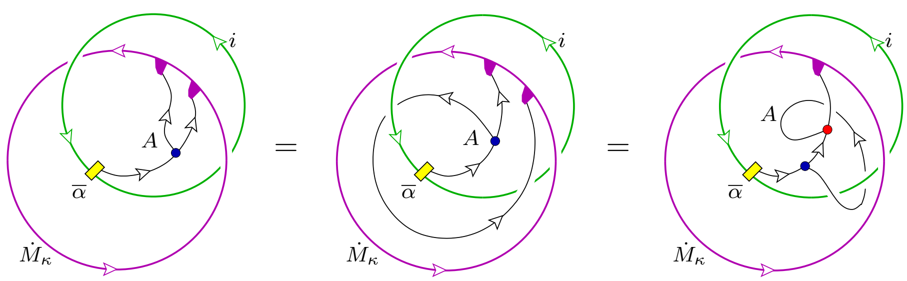
$$= S_{0,0} \sum_{i\in\mathcal{I}} \sum_{\alpha} S^A_{\kappa,i\alpha} \tag{5.98}$$



Here M_κ is a simple A-module, and the second sum extends over the dual basis $\{\bar\alpha\}$ of the morphism space $\mathrm{Hom}(U_i, A\otimes U_i)$. When $X = \dot M_\kappa$ and $\Phi = \rho_\kappa$ is a representation morphism, then the ribbon graph on the right hand side of (5.98) looks like (5.97), except for the fact that α in (5.97) is restricted to be local. This constraint does, however, not pose any problem:

Lemma 5.15 :
The relation (5.98), with $X = \dot M_\kappa$ and $\Phi = \rho_\kappa$, remains true when the sum on the right hand side is restricted to local couplings α.

Proof:
We have the following identities of ribbon graphs in S^3:



(5.99)

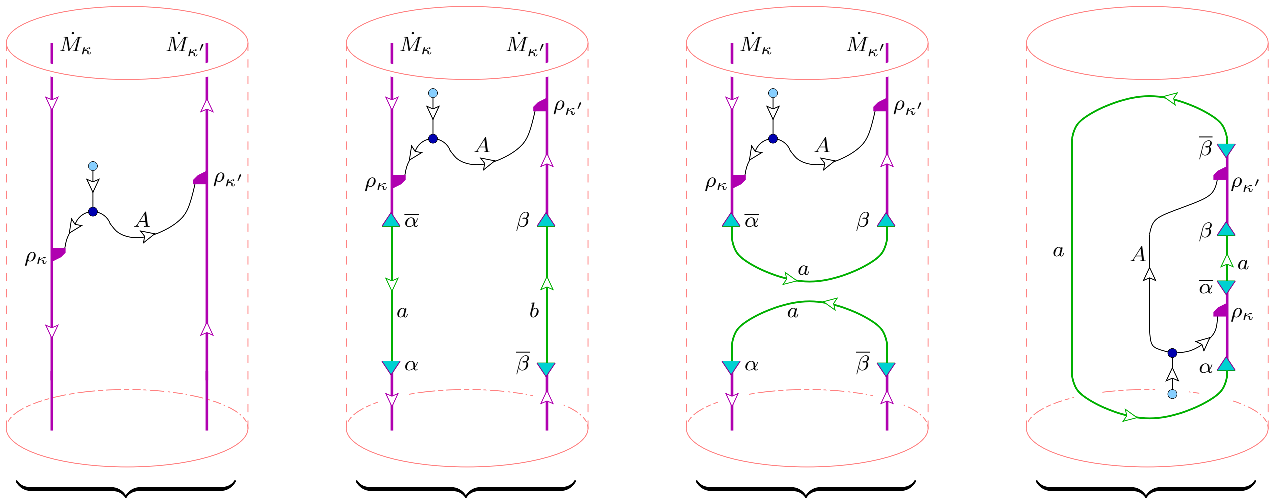
This means that one may always dress the coupling α on the right hand side of (5.98) by the projector P_i. This, in turn, implies that the sum does not change its value when restricted to local couplings. ✓

The following two results imply that, when restricted to a basis of local couplings in $\mathrm{Hom}(A\otimes U_i, U_i)$, S^A is invertible.

Proposition 5.16 :
The matrix S^A is a left-inverse for $\tilde S^A$. That is, for two simple A-modules $M_\kappa, M_{\kappa'}$ one has

$$\sum_{i\in\mathcal{I}} \sum_{\alpha \text{ local}} S^A_{\kappa,i\alpha}\, \tilde S^A_{i\alpha,\kappa'} = \delta_{\kappa,\kappa'} \,. \tag{5.100}$$

Proof:
Consider the defining relation (5.98) for $X = \dot{M}_{\kappa'}$, where $M_{\kappa'}$ is a simple A-module and $\Phi = \rho_{\kappa'}$ the representation morphism. By lemma 5.15, the range of summation on the right hand side of (5.98) can be restricted to local α, thus resulting in the left hand side of (5.100). On the other hand, defining the graphs

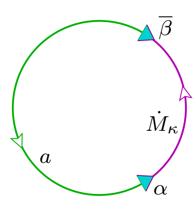


$=: \Gamma_1 \qquad =: \Gamma_2 \qquad =: \Gamma_3 \qquad =: \Gamma_4$ (5.101)

the left hand side of (5.98) becomes

$$D_{\kappa,\kappa'} := \Gamma_1 = \sum_{a,b\in\mathcal{I}} \sum_{\alpha,\beta} \Gamma_2 = \sum_{a\in\mathcal{I}} \sum_{\alpha,\beta} \frac{1}{\dim(U_a)}\, \Gamma_3 = \sum_{a\in\mathcal{I}} \sum_{\alpha,\beta} \frac{1}{\dim(U_a)}\, \Gamma_4 \,. \tag{5.102}$$

Here in the first step bases of the type (4.21) for the morphisms between modules and their subobjects are inserted. The second step uses dominance and the fact that $\dim(\mathcal{H}(m;S^2)) = \delta_{m,0}$ for the one-point blocks on the sphere, implying that only the tensor unit $\mathbf{1}$ survives in the intermediate channel. In the third step the ribbon graph is shifted along the S^1 direction and is shown from a slightly different angle.
One can now use relation (4.33), which applies because M_κ and $M_{\kappa'}$ are simple A-modules. It follows that

$$D_{\kappa,\kappa'} \;=\; \sum_{a\in\mathcal{I}} \sum_{\alpha,\beta} \frac{1}{\dim(U_a)} \frac{\dim(U_a)}{\dim(\dot{M}_\kappa)}\, \delta_{\kappa,\kappa'}\, \delta_{\alpha,\beta}$$  $$=\; \delta_{\kappa,\kappa'} \,, \tag{5.103}$$

where in the last equality it is used that by the completeness property (4.22) the remaining sum over a and α simply amounts to a factor of $\dim(\dot{M}_\kappa)$. ✓

Proposition 5.17:
The matrix $\tilde{S}^A$ has a right-inverse. That is, there exists a matrix $\Sigma = (\Sigma_{\kappa,j\beta})$, with $\kappa \in \mathcal{J}$, $j \in \mathcal{I}$

and β local in $\mathrm{Hom}(A\otimes U_j,U_j)$, such that

$$\sum_{\kappa\in\mathcal{J}} \tilde{S}^A_{i\alpha,\kappa}\,\Sigma_{\kappa,j\beta} = \delta_{i,j}\,\delta_{\alpha,\beta}\,. \tag{5.104}$$

Proof:
According to lemma 5.14 the matrices L^A and $\tilde{L}^A$ are each others' inverses:

$$\sum_{j\in\mathcal{I}}\sum_{\beta} \tilde{L}^A_{i\alpha,j\beta}\,L^A_{j\beta,k\gamma} = \delta_{i,k}\delta_{\alpha,\gamma}\,. \tag{5.105}$$

Note that the β-summation runs over a basis of the *full* coupling space $\mathrm{Hom}(A\otimes U_j,U_j)$. Let us switch from this basis $\{\beta\}$, which consists of eigenvectors of P_j, to the basis (4.25) of $\mathrm{Hom}(A\otimes U_j,U_j)$ that is labelled by $(\kappa\,\rho\sigma)$ (recall also theorem 4.15). Let $D^j_{\beta,\kappa\,\rho\sigma}$ denote the (invertible) matrix that implements this change of basis, and define

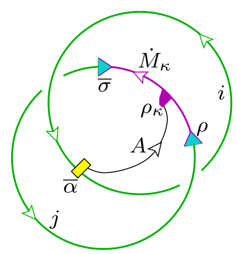
$$K_{(k,\kappa\,\rho\sigma),k\gamma} := \sum_{\beta}(D^j)^{-1}_{\kappa\,\rho\sigma,\beta}\,L^A_{j\beta,k\gamma} \qquad \text{and} \qquad \tilde{K}_{i\alpha,(j,\kappa\,\rho\sigma)} := \sum_{\beta}\tilde{L}^A_{i\alpha,j\beta}\,D^j_{\beta,\kappa\,\rho\sigma}\,. \tag{5.106}$$

The matrices K and $\tilde{K}$ are inverse to each other, too:

$$\sum_{j\in\mathcal{I}}\sum_{\kappa\in\mathcal{J}}\sum_{\rho,\sigma} \tilde{K}_{i\alpha,(j,\kappa\,\rho\sigma)}\,K_{(j,\kappa\,\rho\sigma),k\gamma} = \delta_{i,k}\,\delta_{\alpha,\gamma}\,. \tag{5.107}$$

The pictorial representation of $\tilde{K}$ is

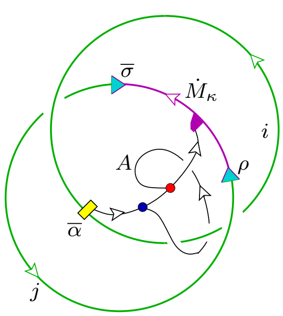
$$\tilde{K}_{i\alpha,(j,\kappa\,\rho\sigma)} \;=$$



(5.108)

We now impose the condition that α is local. We are then allowed to insert a projector P_i in the graph (5.108). This leads to the following identities:

$$\tilde{K}_{i\alpha,(j,\kappa\,\rho\sigma)} \;=$$



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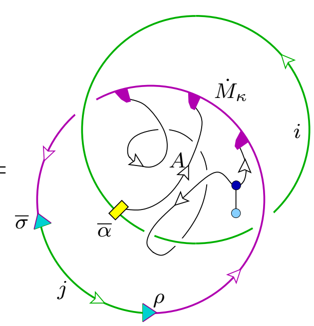

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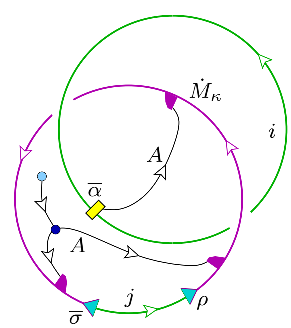


(5.109)

In the last picture we can use the orthogonality relation (4.33) of representation functions (in (4.33), take $k=l$, take the trace on both sides and sum over $\gamma=\delta$). As a consequence, for local α we have the equality

$$\tilde{K}_{i\alpha,(j,\kappa\,\rho\sigma)} = \delta_{\rho,\sigma}\,\frac{\dim(U_j)}{\dim(\dot{M}_\kappa)}\,\tilde{S}^A_{i\alpha,\kappa}\,. \tag{5.110}$$

Using this relation, (5.107) takes the form

$$\sum_{j\in\mathcal{I}}\sum_{\kappa\in\mathcal{J}}\sum_{\rho,\sigma}\delta_{\rho,\sigma}\,\frac{\dim(U_j)}{\dim(\dot M_\kappa)}\,\tilde S^A_{i\alpha,\kappa}K_{(j,\kappa\,\rho\sigma),k\gamma} = \sum_{\kappa\in\mathcal{J}}\tilde S^A_{i\alpha,\kappa}\,\Sigma_{\kappa,k\gamma} = \delta_{i,k}\,\delta_{\alpha,\gamma} \tag{5.111}$$

with

$$\Sigma_{\kappa,k\gamma} := \sum_{j\in\mathcal{I}}\sum_{\rho}\frac{\dim(U_j)}{\dim(\dot M_\kappa)}\,K_{(j,\kappa\,\rho\rho),k\gamma}\,. \tag{5.112}$$

Thus the matrix Σ defined this way, with γ restricted to be local, is right-inverse to $\tilde S^A$. ✓

We conclude that $\tilde S^A$ has both a left- and a right-inverse. Thus the two inverses coincide, i.e. S^A and $\tilde S^A$ are square matrices and inverse to each other. This provides us with a key information for the next section: For $M_\kappa, M_{\kappa'}$ simple A-modules, U_i, U_j simple objects and α,β local basis vectors in the corresponding coupling spaces, we have

$$\sum_{i\in\mathcal{I}}\sum_{\alpha\ \text{local}} S^A_{\kappa,i\alpha}\,\tilde S^A_{i\alpha,\kappa'} = \delta_{\kappa,\kappa'} \qquad \text{and} \qquad \sum_{\kappa\in\mathcal{J}}\tilde S^A_{i\alpha,\kappa}\,S^A_{\kappa,j\beta} = \delta_{i,j}\,\delta_{\alpha,\beta}\,. \tag{5.113}$$

For now let us just note one astonishing consequence of this result.

Theorem 5.18:
Let A be a symmetric special Frobenius algebra in the modular tensor category $\mathcal{C}$. Then

$$|\,\text{isom. classes of simple left } A\text{-modules}\,| = \mathrm{tr}\left[C\,Z(A)\right]. \tag{5.114}$$

Proof:
Consider the space of local couplings in $\mathrm{Hom}(A\otimes U_k,U_k)$, or equivalently, in $\mathrm{Hom}(A\otimes U_k,U_{\bar k}^\vee)$. By lemma 5.6, its dimension is equal to $Z_{k,\bar k}$. Thus a basis B for all local couplings in $\bigoplus_{k\in\mathcal{I}}\mathrm{Hom}(A\otimes U_k,U_k)$ has $\sum_{k\in\mathcal{I}}Z_{k,\bar k}$ elements. Moreover, since S^A is a square matrix, the number of isomorphism classes of simple modules must be the same as the number of basis vectors in B. ✓

Remark 5.19:
(i) In CFT terms, this result tells us that for the CFT based on A the number of elementary boundary conditions (corresponding to simple A-modules) is the same as the number of Ishibashi states, whose total number is $\sum_{k\in\mathcal{I}}Z_{k\bar k}$. This is the completeness condition for boundary conditions [29, 14]. The completeness condition enters in the proof [14] that the annulus coefficients of a CFT furnish a NIM-rep of the fusion rules. Theorem 5.18 implies that CFTs constructed from algebra objects fulfill this condition, and hence their annulus coefficients must provide a NIM-rep. In theorem 5.20 below we will see explicitly that this is indeed the case.
(ii) By combining theorem 5.18 with propositions 4.6 and 5.3 we arrive at the following relation for the category of A-B-bimodules:

$$\begin{aligned}|\,\text{isom. classes of simple } A\text{-}B\text{-bimodules}\,| &= \mathrm{tr}\left[CZ(A{\otimes}B_{\mathrm{op}})\right]\\ &= \mathrm{tr}\left[CZ(A)CZ(B)^{\mathrm{t}}\right]\\ &= \mathrm{tr}\left[Z(A)Z(B)^{\mathrm{t}}\right],\end{aligned} \tag{5.115}$$

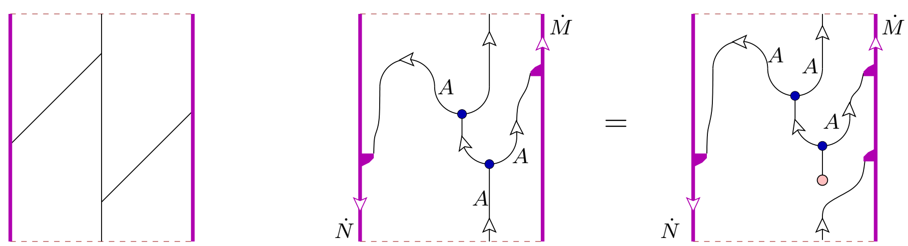
where in the last step we used that the charge conjugation matrix commutes with a modular invariant matrix Z.

Recall the comment in the end of section 4.4 that A-B-bimodules describe tensionless interfaces between the CFTs associated to the algebras A and B. In the special case $A = B$ equation (5.115) reproduces the relation for the number of generalised defect lines found in [34].

5.8 Annulus partition functions

Boundary conditions correspond to modules of a symmetric special Frobenius algebra A, and elementary boundary conditions are simple A-modules. In this section we work out the partition function $\mathrm{A}_M{}^N$ of an annulus with boundary conditions M and N. Let us for now take M and N to be simple A-modules. The world sheet X for the annulus amplitude is an annulus. Its complex double $\hat{\mathrm{X}}$ is a torus $\mathrm{T} = S^1 \times S^1$, and the connecting three-manifold M_X is a full torus $D \times S^1$, where D is a disk.

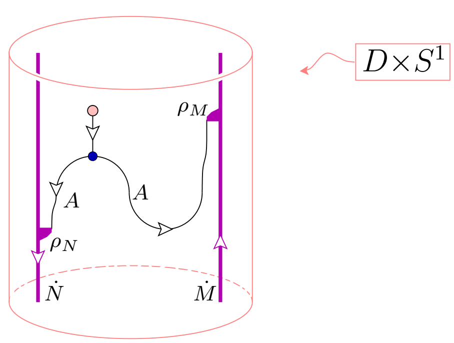
According to the prescription in section 5.1, the first step in the construction of the partition function is to specify a triangulation of the world sheet X; let us choose the one given in the picture on the left hand side of



(5.116)

(Here top and bottom are to be identified, which is indicated by the dashed lines.) Then we substitute the elements (5.6), (5.7) to convert the triangulation into a ribbon graph; simplifying the result, we arrive at the graph in the middle of (5.116). Using the Frobenius property of A and the representation property of ρ_M, this graph can in turn be rewritten as on the right hand side. It follows that the A-ribbon that runs around the annulus can be removed. Then the ribbon graph describing the annulus amplitude finally becomes

$\mathrm{A}_M{}^N \; =$



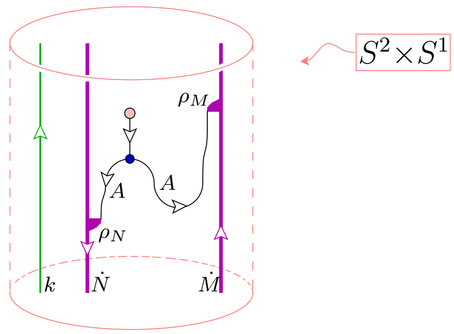
(5.117)

This graph determines an element $\mathrm{A}_M{}^N$ in the space $\mathcal{H}(\emptyset; \mathrm{T})$ of zero-point conformal blocks on the torus. To obtain the annulus coefficients we must expand $\mathrm{A}_M{}^N$ in a basis of blocks on the torus, i.e. in terms of characters (5.15):

$$\mathrm{A}_M{}^N = \sum_{k \in \mathcal{I}} \mathrm{A}_{kM}{}^N \, |\chi_k; \mathrm{T}\rangle \,. \tag{5.118}$$

Just like for the torus partition function, we can isolate the coefficient $\mathrm{A}_{kM}{}^{N}$ by gluing the basis dual to $|\chi_k;\mathrm{T}\rangle$ to the boundary of the three-manifold on both sides of the equation. We are then left with

$$\mathrm{A}_{kM}{}^{N} \;= \tag{5.119}$$



We now establish that the quantities defined by formula (5.117) indeed possess the properties that befit a partition function on an annulus.

Theorem 5.20:
Let A be a symmetric special Frobenius algebra, $M\equiv M_\kappa$ and $N\equiv M_{\kappa'}$ ($\kappa,\kappa'\in\mathcal{J}$) simple A-modules, and U_k ($k\in\mathcal{I}$) a simple object. Then the numbers $\mathrm{A}_{kM}{}^{N}$ given by the invariant of the ribbon graph (5.119) have the following properties:
(i) Non-negativity:

$$\mathrm{A}_{kM}{}^{N} \in \mathbb{Z}_{\geq 0}\,. \tag{5.120}$$

(ii) Uniqueness of the vacuum:

$$\mathrm{A}_{0M}{}^{N} = \delta_{M,N}\,. \tag{5.121}$$

(iii) Exchange of boundary conditions:

$$\mathrm{A}_{k\,M}{}^{N} = \mathrm{A}_{\bar k\,N}{}^{M}\,. \tag{5.122}$$

(iv) Consistency with the bulk spectrum:

$$\mathrm{A}_{kM}{}^{N} = \sum_{p\in\mathcal{I}}\;\sum_{\alpha\ \text{local}} S^A_{N,p\alpha}\,\frac{S_{kp}}{S_{0p}}\,\tilde S^A_{p\alpha,M}\,. \tag{5.123}$$

(v) NIM-rep of the fusion rules:

$$\mathrm{A}_i\,\mathrm{A}_j = \sum_{k\in\mathcal{I}} N_{ij}{}^{k}\,\mathrm{A}_k\,, \tag{5.124}$$

where the A_k are the matrices with entries $\mathrm{A}_{kM}{}^{N}$.

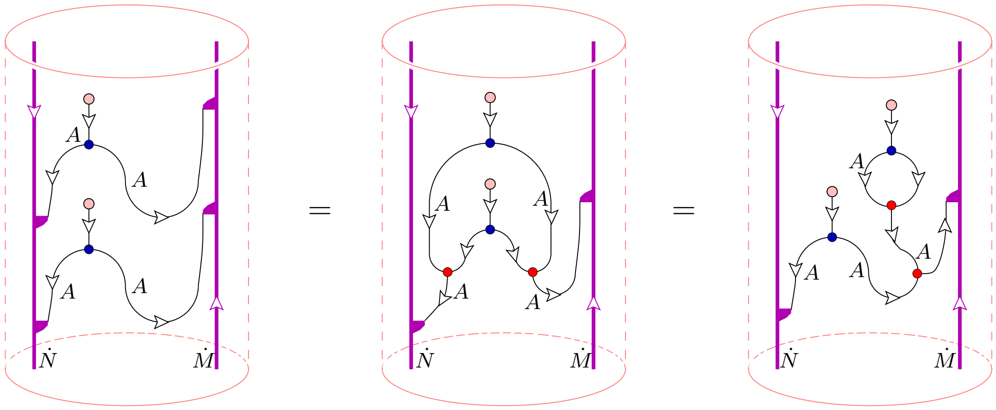
Proof:
To show (i) one cuts the three-manifold (5.119) along an S^2 to obtain a ribbon graph in $S^2\times[-1,1]$. This defines a linear map

$$Q_k:\quad \mathcal{H}(k,(\dot N,-),\dot M;S^2)\rightarrow\mathcal{H}(k,(\dot N,-),\dot M;S^2)\,, \tag{5.125}$$

from which the annulus coefficients are recovered via

$$\mathrm{A}_{kM}{}^{N} = \mathrm{tr}_{\mathcal{H}(k,(\dot N,-),\dot M;S^2)}Q_k\,. \tag{5.126}$$

Next one considers the identities



(5.127)

Here the first step uses the representation property, while the second step results from repeated application of the associativity and Frobenius property of A. Using also that A is special, as well as the defining property of the unit, one concludes that the map Q_k is a projector,

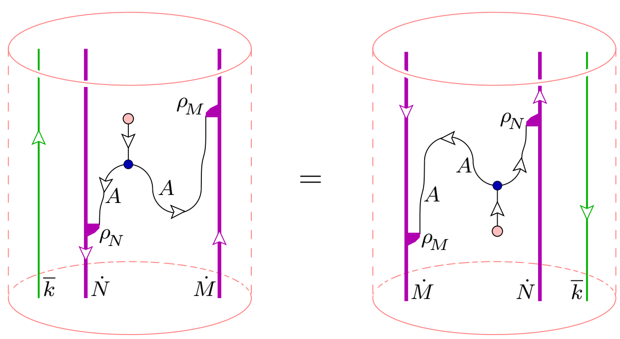
$$Q_k \circ Q_k = Q_k \,, \tag{5.128}$$

and therefore

$$\mathrm{tr}_{\mathcal{H}(k,(\dot N,-),\dot M;S^2)} \in \mathbb{Z}_{\ge 0} \,. \tag{5.129}$$

(ii): For $k=0$, $\mathrm{A}_{kM}{}^N$ is given by the graph Γ_1 in picture (5.7). The statement of (ii) is therefore equivalent to the result (5.103) that was obtained in the course of proving proposition 5.16.

To see (iii) it is sufficient to 'rotate' the ribbon graph (5.119) according to

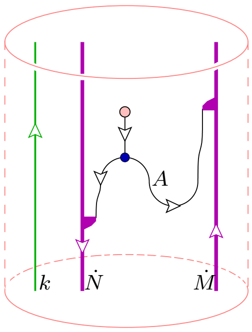
$\mathrm{A}_{\bar k\, M}{}^N \;=$ 

(5.130)

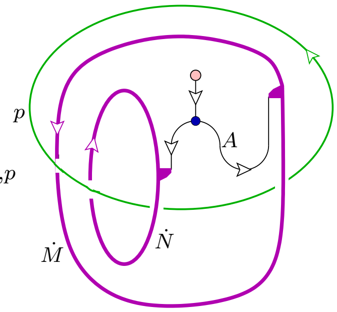
The second equality holds because the two three-manifolds with embedded ribbon graphs are related by an (invertible) orientation preserving homeomorphism and thus have the same topological invariant. The second ribbon graph in (5.130) equals $\mathrm{A}_{k\,N}{}^M$, as can be seen by using that A is symmetric and by inserting a pair $\pi_k \circ \pi_k^{-1}$ to replace the $\bar k$-ribbon by a k-ribbon with reversed orientation of the core.

For the proof of (iv) we use the surgery relation (5.20) with $i=k$ and $X=\dot N^\vee \otimes \dot M$, and with

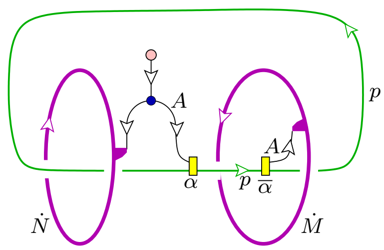
Φ corresponding to an A-ribbon suspended between $N^\vee$ and M as in (5.119). This yields



$$= \; S_{0,0} \sum_{p\in\mathcal{I}} S_{k,p}$$

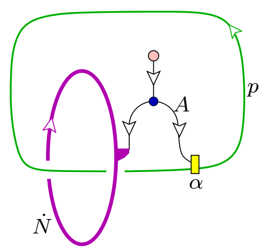


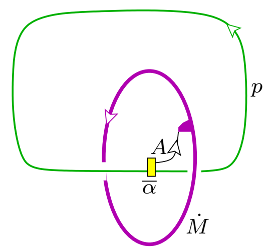
$$= \; S_{0,0} \sum_{p\in\mathcal{I}} S_{k,p} \sum_{\alpha}$$



(5.131)

$$= \; S_{0,0} \sum_{p\in\mathcal{I}} \frac{S_{k,p}}{\dim(U_p)} \sum_{\alpha \text{ local}}$$

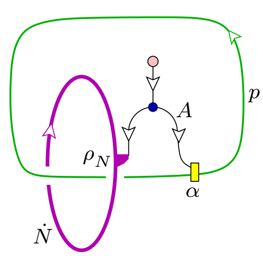




Here the first step is the relation (5.20). In the second step dominance is used to insert a basis for $\mathrm{Hom}(A{\otimes}U_p, U_p)$ (in the summation over intermediate objects U_m, only U_p survives, as is seen by reading the picture from left to right, which involves a morphism from $\mathbf{1}$ to $U_m{\otimes}U_p^\vee$ to $\mathbf{1}$, implying that $m\,{=}\,p$). The index α runs over the eigenbasis in $\mathrm{Hom}(A{\otimes}U_p, U_p)$, and $\bar\alpha$ denotes the dual basis. In the last step, dominance is used once more, and again only a single intermediate channel contributes, this time the tensor unit $\mathbf{1}$. Moreover, the sum over α can be restricted to only local basis vectors of $\mathrm{Hom}(A{\otimes}U_p, U_p)$, by the same reasoning that was used in the proof of lemma 5.15, i.e. by similar moves as in (5.99).

The second ribbon graph in the last line of (5.131) is nothing but the quotient $\tilde S^A_{p\alpha,M}/S_{0,0}$, as defined in (5.97). Consider now the result (5.131) for the special case $k\,{=}\,0$. Then according to (5.121) we must get $\delta_{M,N}$. Using also that $S_{0,p}/\dim(U_p)\,{=}\,S_{0,0}$, it follows that the first ribbon graph in the last line describes the matrix that is inverse to $\tilde S^A$. Thus it equals $S^A_{N,p\alpha}/S_{0,0}$, with S^A as defined in (5.98). In pictures,

$$S^A_{N,p\alpha} \;=\; S_{0,0}$$



(5.132)

This completes the proof of part (iv).

Finally, property (v) is a direct consequence of (iv). Indeed, substituting (5.123) into the right hand side of (5.124), we get

$$\begin{aligned}\sum_{R\in\mathcal{J}} \mathrm{A}_{iM}{}^{R}\mathrm{A}_{jR}{}^{N} &= \sum_{R\in\mathcal{J}}\sum_{p,\alpha} S^A_{R,p\alpha}\,\frac{S_{p,i}}{S_{p,0}}\,\tilde S^A_{p\alpha,M}\sum_{q,\beta} S^A_{N,q\beta}\,\frac{S_{q,j}}{S_{q,0}}\,\tilde S^A_{q\beta,R}\\ &= \sum_{p,\alpha}\sum_{q,\beta} S^A_{N,q\beta}\,\frac{S_{p,i}}{S_{p,0}}\,\delta_{p,q}\,\delta_{\alpha,\beta}\,\frac{S_{q,j}}{S_{q,0}}\,\tilde S^A_{p\alpha,M}\\ &= \sum_{p,\alpha}\sum_{k\in\mathcal{I}} \frac{S_{p,i}S_{p,j}S^{-1}_{p,k}}{S_{p,0}}\,S^A_{N,p\alpha}\,\frac{S_{p,k}}{S_{p,0}}\,\tilde S^A_{p\alpha,M} \;=\; \sum_{k\in\mathcal{I}} N_{ij}{}^{k}\,\mathrm{A}_{kM}{}^{N}\,,\end{aligned} \tag{5.133}$$

which establishes that we are indeed dealing with a NIM-rep. ✓

Remark 5.21:
(i) For $A=\mathbf{1}$, i.e. for the charge conjugation modular invariant $Z_{ij}=\delta_{i,\bar\jmath}$, one recovers [51] the original result [73] that in this situation the annulus coefficients coincide with the fusion rules.
(ii) The matrices A_k commute and are normal (since $\mathrm{A}_k^{\mathrm{t}}=\mathrm{A}_{\bar k}$, see formula (5.122)) and hence can be simultaneously diagonalised, with eigenvalues $S_{k,a}/S_{0,a}$. Requiring consistency between the open and closed string channel in the annulus amplitude leads to the condition that the multiplicity $m(a)$ of the set $\{S_{k,a}/S_{0,a}\,|\,k\in\mathcal{I}\}$ of eigenvalues must be equal to $Z_{a\bar a}$. This condition has been thoroughly investigated in [21], where many examples of modular invariants without associated NIM-rep as well as NIM-reps without associated modular invariant were found.
Since the number of basis elements in the space $\mathrm{Hom}_{\mathrm{loc}}(A\otimes U_p,U_p)$ of local couplings is $Z_{p\bar p}$ (see lemma 5.6), part (iv) of theorem 5.20 implies that for any CFT obtained from an algebra object, the annulus coefficients $\mathrm{A}_{kM}{}^{N}$ are consistent with the modular invariant Z_{ij} in the sense of definition 4 of [21].
(iii) In [35], polynomial equations and trace identities for the (integral) coefficients appearing in the torus, cylinder, Möbius and Klein bottle amplitudes have been presented. Those identities of [35] that involve Z_{ij} and $\mathrm{A}_{kM}{}^{N}$ follow from completeness of the boundary conditions, and are thus automatically satisfied by CFTs constructed from algebra objects. For example, one derives the trace identity [35]

$$\begin{aligned}\sum_M \mathrm{A}_{kM}{}^{M} &= \sum_M\sum_{p\in\mathcal{I}}\sum_{\alpha\ \text{local}} S^A_{M,p\alpha}\,\frac{S_{k,p}}{S_{0,p}}\,\tilde S^A_{p\alpha,M} = \sum_{p\in\mathcal{I}}\sum_{\alpha\ \text{local}}\frac{S_{k,p}}{S_{0,p}}\\ &= \sum_{p\in\mathcal{I}} Z_{p\bar p}\,\frac{S_{k,p}}{S_{0,p}} = \sum_{i,j,p\in\mathcal{I}} S_{p,i}\,Z_{ij}\,S^{-1}_{\bar p,j}\,\frac{S_{k,p}}{S_{0,p}} = \sum_{i,j,p\in\mathcal{I}} N_{ik}{}^{\bar\jmath}\,Z_{ij}\,.\end{aligned} \tag{5.134}$$

Here we used the facts that S^A and $\tilde S^A$ are inverse to each other and that Z commutes with S.

Boundary fields will be labelled by (M,N,k,α), where M,N are A-modules, U_k is a simple object and α an element of $\mathrm{Hom}_A(M\otimes U_k,N)$. That this prescription is consistent with the annulus amplitude $\mathrm{A}_{kM}{}^{N}$ follows from

Proposition 5.22 :
Let U_k $(k \in \mathcal{I})$ be a simple object and M, N two (not necessarily simple) A-modules of a symmetric special Frobenius algebra A. Then

$$\mathrm{A}_{kM}{}^{N} = \dim\big[\mathrm{Hom}_A(M \otimes U_k, N)\big] , \tag{5.135}$$

where $\mathrm{A}_{kM}{}^{N}$ is the invariant associated to the ribbon graph (5.119).

Before proving the proposition, it is helpful to introduce some additional notation. Define the linear map Q_{kM}^{N} acting on $\mathrm{Hom}(\dot M \otimes U_k, \dot N)$ via $Q_{kM}^{N}(\Phi) := \overline{\Phi}$, where $\overline{\Phi}$ denotes the A-averaged morphism as introduced in formula (4.11). By combining parts (i) and (ii) of lemma 4.4 it follows that Q_{kM}^{N} is a projector and that

$$\mathrm{Im}\, Q_{kM}^{N} = \mathrm{Hom}_A(M \otimes U_k, N) \subseteq \mathrm{Hom}(\dot M \otimes U_k, \dot N) . \tag{5.136}$$

Let us choose an eigenbasis $\{\psi_\alpha^{kMN}\}$ of Q_{kM}^{N}, i.e.

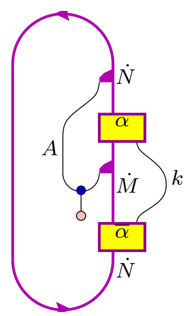
$$\{\psi_\alpha^{kMN}\} \subset \mathrm{Hom}(\dot M \otimes U_k, \dot N) \quad \text{such that} \quad Q_{kM}^{N} \psi_\alpha^{kMN} = \varepsilon_\alpha \psi_\alpha^{kMN} \quad \text{with } \varepsilon_\alpha \in \{0, 1\} . \tag{5.137}$$

Further, we fix a basis $\{\bar\psi_\alpha^{kMN}\}$ in $\mathrm{Hom}(\dot N, \dot M \otimes U_k)$ that is dual to $\{\psi_\alpha^{kMN}\}$ in the sense that

$$\mathrm{tr}\left[\psi_\alpha^{kMN} \circ \bar\psi_\beta^{kMN}\right] = \delta_{\alpha,\beta} . \tag{5.138}$$

Then we can write

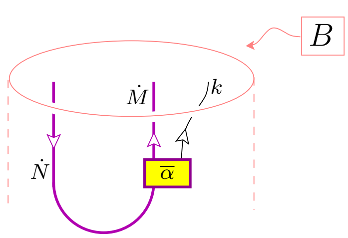
$$\mathrm{tr}\, Q_{kM}^{N} = \sum_\alpha \mathrm{tr}\left[Q_{kM}^{N}(\psi_\alpha^{kMN}) \circ \bar\psi_\alpha^{kMN}\right] = \sum_\alpha \tag{5.139}$$

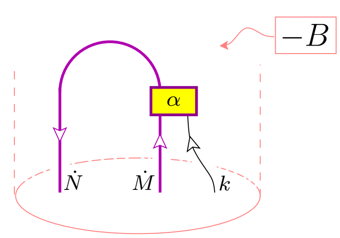


where the first trace is a trace over the vector space $\mathrm{Hom}(\dot M \otimes U_k, \dot N)$, while the second trace is a trace in the category theoretic sense of (2.14).

Proof of proposition 5.22:
Consider the two ribbon graphs

$$\Gamma_1 := \qquad\qquad \Gamma_2 := \tag{5.140}$$



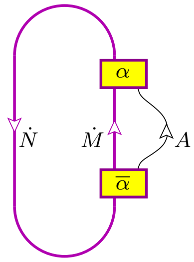


The three-manifolds in which these ribbon graphs are embedded are both solid three-balls, but with opposite orientation, $\pm B$. In the drawing, the disk bounded by the circle indicates the boundary S^2 of $\pm B$. Define elements

$$\begin{aligned} v_\alpha &:= Z(\Gamma_1, \emptyset, S^2)\, 1 && \in \mathcal{H}((\dot N, -), \dot M, k; S^2) , \\ \bar v_\alpha &:= (S_{0,0} \dim(\dot N))^{-1}\, Z(\Gamma_2, S^2, \emptyset) && \in \big(\mathcal{H}((\dot N, -), \dot M, k; S^2)\big)^* . \end{aligned} \tag{5.141}$$

That indeed $\bar{v}_\alpha(v_\beta) = \delta_{\alpha,\beta}$ can be seen as follows:

$$\bar{v}_\alpha(v_\beta) \;=\; (S_{0,0}\dim(\dot{N}))^{-1}\, S_{0,0}$$

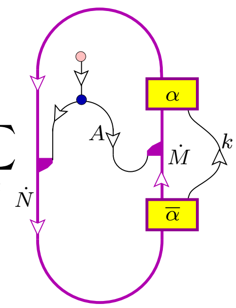


$$=\; \delta_{\alpha,\beta} \tag{5.142}$$

In the first equality the concatenation of $\bar{v}$ and v is expressed as a ribbon graph in S^3 obtained by glueing the two three-balls along their boundary. Recall the convention described below (5.13), introducing a factor $S_{0,0}$ for the invariant associated to a ribbon graph in S^3.

Having established that the vectors v_α and $\bar{v}_\beta$ constitute dual bases of $\mathcal{H}((\dot{N},-),\dot{M},k;S^2)$ and $\mathcal{H}((\dot{N},-),\dot{M},k;S^2)^*$, we can rewrite the invariant (5.119) as in (5.126) and express the trace as (we can move the k-ribbon in (5.119) from the left side of the module ribbons to the right side)

$$\mathrm{A}_{kM}{}^{N} = \mathrm{tr}_{\mathcal{H}((\dot{N},-),\dot{M},k;S^2)} Q_k = \sum_\alpha \bar{v}_\alpha(Q_k v_\alpha) = \frac{S_{0,0}}{S_{0,0}\dim(\dot{N})} \sum_\alpha$$



$$= \mathrm{tr}\, Q_{kM}^{N} \tag{5.143}$$

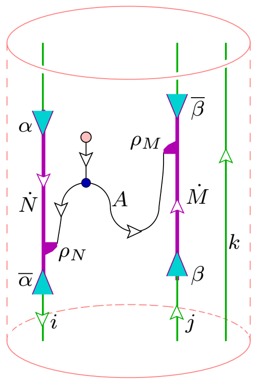
To see the last equality, we note that the ribbon graph above can be transformed into the one shown in figure (5.139) (using that the algebra A is symmetric). Equation (5.136) now implies the proposition. ✓

5.9 The case $N_{ij}{}^{k} \in \{0,1\}$ and $\dim \mathrm{Hom}(U_k, A) \in \{0,1\}$

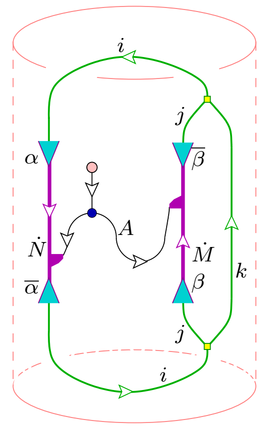
This is the last part of our meta example (as far as the present paper is concerned). We will illustrate how to compute the invariant associated to the annulus partition function (5.119).

This can be done by the following series of transformations:

$$\mathrm{A}_{kM}{}^{N} \;=\; \sum_{i,j\in\mathcal{I}}\;\sum_{\alpha,\beta}$$

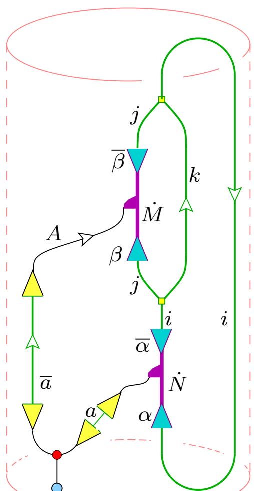


$$=\; \sum_{i,j\in\mathcal{I}}\;\sum_{\alpha,\beta}\;\frac{1}{\dim(U_i)}$$

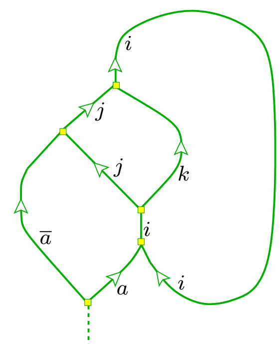


(5.144)

$$=\; \sum_{\substack{i,j,a\in\mathcal{I}\\ \alpha,\beta}}\;\frac{1}{\dim(U_i)}$$



$$=\; \sum_{\substack{i,j,a\\ \alpha,\beta}}\;\frac{\Delta_0^{\bar a a}}{\dim(U_i)}\;\rho^{M\,j\beta}_{\bar a\,,\,j\beta}\;\rho^{N\,i\alpha}_{a\,,\,i\alpha}$$



Here the usual calculational devices are used: In the first step bases for the morphisms involving the simple A-modules M and N are inserted. The second step uses dominance twice, together with $\dim(\mathcal{H}((i,-),m;S^2))=\delta_{m,i}$ and $\dim(\mathcal{H}(m;S^2))=\delta_{m,0}$, which imply that that the sums over intermediate simple objects each reduce to a single term. The third step consists again in the insertion of a basis, and in the fourth step one substitutes the definition (4.61) for ρ.

To arrive at our final formula, we also use an inverse fusion move on the last graph above, together with relation (2.60). The result is

$$\mathrm{A}_{kM}{}^{N} = \sum_{a\prec A}\sum_{i,j\in\mathcal{I}}\sum_{\alpha,\beta} \rho^{N\,i\alpha}_{a\,,\,i\alpha}\rho^{M\,j\beta}_{\bar a\,,\,j\beta}\,\Delta_0^{\bar a a}\cdot \mathsf{G}^{\,(\bar a\,j\,k)\,i}_{\;j\,i}\,\mathsf{F}^{\,(\bar a\,a\,i)\,i}_{\;i\,0}\,, \tag{5.145}$$

where α labels a basis of $\mathrm{Hom}(U_i,\dot M)$ and β a basis of $\mathrm{Hom}(U_j,\dot N)$. Here the coproduct $\Delta_0^{\bar a a}$ can also be expressed through the multiplication using (3.83).

5.9.1 Example: Free boson

Substituting the free boson modular data (2.77), the A_{2r}-algebra (3.84), (3.85) and the expressions (4.79) for the representation matrices into the general formula (5.145), we obtain

$$A_{[k]\,M_m}{}^{M_n} = \frac{r}{N}\sum_{j\in\mathcal{I}}\delta_{[j+k-m]\prec A}\,\delta_{[j-n]\prec A} = \delta_{n+k,m \bmod 2r}\,. \tag{5.146}$$

Note that when the two boundary conditions are equal, then the result no longer depends on $m=n$. Thus each of the $2r$ distinct elementary boundary conditions has the same field content.

5.9.2 Example: E_7 modular invariant

As seen in section 4.5.2, the algebra object A that gives the E_7 modular invariant of the $\mathfrak{su}(2)_{16}$ WZW model has seven isomorphism classes of simple modules. We label representatives for these classes as

$$\begin{aligned} &M_1 = A \;, \quad M_2 = \mathrm{Ind}_A(1)\,, \quad M_3 = \mathrm{Ind}_A(2)\,, \quad M_4 = \mathrm{Ind}_A(3)\,, \\ &M_5 = P\,, \quad M_6 = Q\,, \quad M_7 = R\,. \end{aligned} \tag{5.147}$$

(Also recall from section 4.5.2 that the latter three are not induced modules.) We can now numerically evaluate formula (5.145) for the annulus coefficients $\mathrm{A}_{kM}{}^{N}$. To obtain the representation matrices we proceed as described in section 4.5.2 and use formula (4.75). One then directly verifies that the numbers $\mathrm{A}_{kM}{}^{N}$ are non-negative integers, satisfy $\mathrm{A}_{0M}{}^{N}=\delta_{M,N}$, and furnish a NIM-rep of the fusion rules.

To make contact with the classification of boundary conditions in [129, 130] we also present the matrix $\mathrm{A}_{(1)}$:

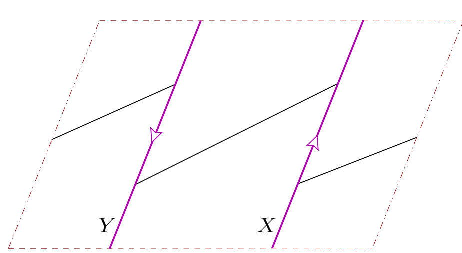
$$\mathrm{A}_{(1)} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} \tag{5.148}$$

(Rows and columns are ordered according to the labelling (5.147), and zero entries have been replaced by dots to improve readability.) In agreement with the results of [129, 130], this is indeed the adjacency matrix of the Dynkin diagram of the Lie algebra E_7.

5.10 Defect lines and double fusion algebra

In this section we consider the partition function of a torus with two defect lines inserted – a setup studied in [34]. In particular we recover the property that the coefficients of such partition functions furnish a NIM-rep of the 'double fusion algebra', a structure also seen in [11, 123].

As already mentioned at the end of section 4.4 and in remark 5.19, in our framework defect lines are described (and labelled) by A-A-bimodules. Consider the situation where the world sheet is a two-torus T without field insertions, but with two defect lines X, Y running parallel and with opposite orientation. The setup is shown in the following picture, together with the triangulation we will use (compare also (5.23)):



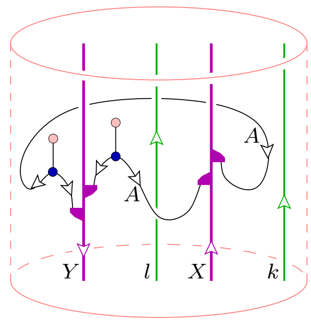
(5.149)

Proceeding similarly to section 5.3 we can express the coefficients $Z_{kl}^{X|Y}$ in the twisted partition function

$$Z^{X|Y} = \sum_{k,l\in\mathcal{I}} Z_{kl}^{X|Y} |\chi_k;\mathrm{T}\rangle \otimes |\chi_l;-\mathrm{T}\rangle \tag{5.150}$$

as the invariant of a ribbon graph. We obtain

$$Z_{kl}^{X|Y} \;=\; \tag{5.151}$$



Before proceeding to prove some properties of these numbers it is useful to slightly change the notation for annulus coefficients, so as to make the dependence on the algebra explicit. Thus for a symmetric special Frobenius algebra B and for left B-modules M, N, let $\mathrm{A}(B)_{kM}^{\;\;\;N}$ denote the number defined in (5.119), but with all algebra ribbons labelled by B.

Theorem 5.23:
Let A be a symmetric special Frobenius algebra, let U_k, U_l be simple objects and X, Y be A-A-bimodules. Abbreviate $B = A\otimes A^{(-1)}$ and $\tilde{B} = A^{(1)}\otimes A$. The numbers $Z_{kl}^{X|Y}$ have the following properties:

(i) $\quad Z_{kl}^{X|Y} \in \mathbb{Z}_{\geq 0}\,.$ (5.152)

(ii) $\quad Z_{kl}^{A|A} = Z(A)_{kl}\,.$ (5.153)

(iii) $\quad Z_{\bar{k}\bar{l}}^{X|Y} = Z_{kl}^{Y|X}\,.$ (5.154)

(iv) $\quad Z_{k0}^{X|Y} = \mathrm{A}(B)_{k\,f(X)}^{\;\;\;\;\;\;f(Y)} \quad$ and $\quad Z_{0l}^{X|Y} = \mathrm{A}(\tilde{B})_{l\,\tilde{f}(X)}^{\;\;\;\;\;\;\tilde{f}(Y)}\,.$ (5.155)

(v) $\quad Z_{kl}^{X|Y} = \sum_R Z_{k0}^{X|R} Z_{0l}^{R|Y} \quad$ or, as matrix equation, $\quad Z_{kl} = Z_{k0}\,Z_{0l}\,.$ (5.156)

(vi) $\quad$ As matrix equations, $\;[Z_{l0}, Z_{k0}] = [Z_{0l}, Z_{0k}] = [Z_{0l}, Z_{k0}] = 0\,.$ (5.157)

(vii) $\quad$ As matrix equation, $\quad Z_{ij}\,Z_{kl} = \sum_{r,s\in\mathcal{I}} N_{ik}^{\;\;r}\,N_{jl}^{\;\;s}\,Z_{rs}\,.$ (5.158)

In (ii), $Z(A)_{kl}$ denotes the coefficients of the untwisted torus partition function (5.30). In (v)–(vii) Z_{kl} is understood as a matrix with entries $(Z_{kl})_{X,Y} = Z_{kl}^{X|Y}$. The sum in (v) is over (representatives of isomorphism classes of) simple A-A-bimodules R. The notation $A^{(n)}$ was defined in (3.65). In (iv), f and $\tilde{f}$ are the isomorphisms defined in (4.19) taking A-A-bimodules to left $A\otimes A^{(-1)}$- and $A^{(1)}\otimes A$-modules, respectively.

Proof:
Property (i) follows along the same lines as the proof of theorem 5.1(ii). That is, the invariant

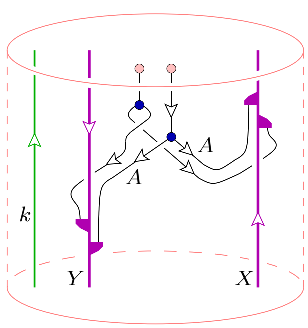
(5.151) can be rewritten as the trace of a projector. The projector is again obtained by cutting the three-manifold in (5.151) along a 'horizontal' S^2; the projector property follows from the representation property of the bimodules X, Y and the properties of A.

To see (ii), note that the ribbon graph resulting from (5.151) when replacing X and Y by A is almost identical to the graph (5.30), except for an additional A-ribbon running vertically. The latter can be removed by transformations similar to (5.116) with M, N set to A (that the geometry in that situation is actually a cylinder, rather than a torus, does not play a role in the calculation).

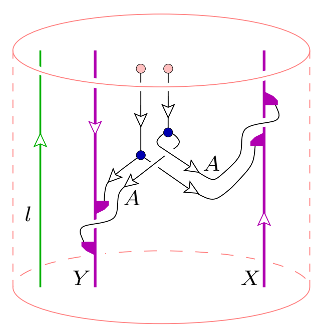
The proof of (iii) uses the same argument as the proof of theorem 5.20(iii): The ribbon graph for $Z_{\bar k \bar l}^{X|Y}$ is that of $Z_{kl}^{Y|X}$ turned upside down (this is a rotation, preserving the orientation of the three-manifold) and with the k- and l-ribbons replaced by $\bar k$- and $\bar l$-ribbons with opposite orientation of their cores.

To show (iv) we draw the ribbon graphs for the annulus coefficients appearing in (5.155), inserting the relation (4.19) between A-A-bimodules and $A{\otimes}A^{(-1)}$- (respectively left $A^{(1)}{\otimes}A$-) modules and the definition of the algebras $A^{(\pm1)}$. The resulting graphs are:

$\mathrm{A}(A{\otimes}A^{(-1)})_{k\,f(X)}^{\ \ \ \ \ f(Y)}$ $=$



$\mathrm{A}(A^{(1)}{\otimes}A)_{l\,\tilde f(X)}^{\ \ \ \ \ \tilde f(Y)}$ $=$



(5.159)

(Here all representation morphisms are those of A-A-bimodules, and all comultiplications are given by the coproduct of A.) In the case of $\mathrm{A}(A{\otimes}A^{(-1)})_{f(X)}^{\ \ \ f(Y)}$, the ribbon graph is obviously equal to the graph (5.151) with $l=0$. For $\mathrm{A}(A^{(1)}{\otimes}A)_{\tilde f(X)}^{\ \ \ \tilde f(Y)}$, the required moves are slightly more complicated; they are again best visualised by using actual ribbons. The main step is to recall that the 'horizontal' direction is an S^2, which allows us to move one of the 'horizontal' A-ribbons around the S^2 in such a way that in the pictorial representation the A-ribbons now seem to wrap around the l-ribbon. Once this step is performed, equality with $Z_{0l}^{X|Y}$ is easily established.

Proof of (v): Let $B = A{\otimes}A^{(-1)}$. Given a simple object U_k and a left B-module M, the tensor product $M{\otimes}U_k$ is again a left B-module (compare the discussion of module categories in section 4.1). By proposition 4.13 the category of left B-modules is semisimple, and hence we can decompose $M{\otimes}U_k$ in terms of simple left B-modules as

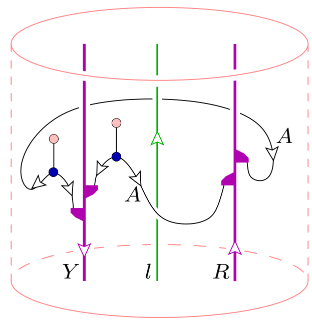
$$M \otimes U_k \;\cong\; \bigoplus_{i\in\mathcal{I}_B} \langle M{\otimes}U_k, S_i\rangle_B \, S_i \,, \tag{5.160}$$

as an isomorphism of left B-modules. Here the sum is over representatives S_i of isomorphism classes of simple B-modules. The multiplicities $\langle M{\otimes}U_k, S\rangle_B$ have been computed in proposition 5.22 to be given by annulus coefficients:

$$\langle M{\otimes}U_k, S\rangle_B = \mathrm{A}(B)_{kM}^{\ \ \ S} \,. \tag{5.161}$$

We can apply this relation to (5.151) by understanding the A-A-bimodule Y as a left B-module $f(Y)$. This results in

$$Z^{X|Y}_{kl} = \sum_R \mathrm{A}(B)_{k\, f(X)}^{\;\;f(R)}$$



$$= \sum_R Z^{X|R}_{k0}\, Z^{R|Y}_{0l} \tag{5.162}$$

The first step makes use of relations (5.160) and (5.161); the sum is over representatives of isomorphism classes of simple A-A-bimodules, which are taken to simple left B-modules via f. The second step inserts definition (5.151) for the ribbon graph and the previous result (5.155). That the first two commutators in part (vi) vanish follows directly from (iv) together with the NIM-rep property (5.124). The last equality is obtained by combining (iii) and (v). Indeed, by (v) we have

$$Z^{X|Y}_{kl} = \sum_R Z^{X|R}_{k0} Z^{R|Y}_{0l} \tag{5.163}$$

as well as, using (iii) and $\bar 0 = 0$,

$$Z^{Y|X}_{\bar k \bar l} = \sum_R Z^{Y|R}_{\bar k 0} Z^{R|X}_{0 \bar l} = \sum_R Z^{X|R}_{0l} Z^{R|Y}_{k0}. \tag{5.164}$$

Employing (iii) once more we see that (5.163) and (5.164) are equal, thereby establishing (vi). Finally, (vii) follows by a short calculation from (iv), (v) and (vi). With $B = A \otimes A^{(-1)}$ and $\tilde B = A^{(1)} \otimes A$ we have

$$\begin{aligned}(Z_{ij} Z_{kl})^{X|Y} &= (Z_{i0} Z_{0j} Z_{k0} Z_{0l})^{X|Y} = (Z_{i0} Z_{k0} Z_{0j} Z_{0l})^{X|Y} = \sum_{R,S,T} Z^{X|R}_{i0} Z^{R|S}_{k0} Z^{S|T}_{0j} Z^{T|Y}_{0l} \\ &= \sum_{R,S,T} \mathrm{A}(B)_{i\, f(X)}^{\;\;f(R)}\, \mathrm{A}(B)_{k\, f(R)}^{\;\;f(S)}\, \mathrm{A}(\tilde B)_{j\, \tilde f(S)}^{\;\;\tilde f(T)}\, \mathrm{A}(\tilde B)_{l\, \tilde f(T)}^{\;\;\tilde f(Y)} \\ &= \sum_{r,s \in \mathcal{I}} \sum_S N_{ik}^{\;\;r}\, \mathrm{A}(B)_{r\, f(X)}^{\;\;f(S)}\, N_{jl}^{\;\;s}\, \mathrm{A}(\tilde B)_{s\, \tilde f(S)}^{\;\;\tilde f(Y)} \\ &= \sum_{r,s \in \mathcal{I}} N_{ik}^{\;\;r} N_{jl}^{\;\;s} \sum_S Z^{X|S}_{r0}\, Z^{S|Y}_{0s}\,.\end{aligned} \tag{5.165}$$

In the next-to-last step it is used that the annulus coefficients furnish a NIM-rep, see theorem 5.20. ✓

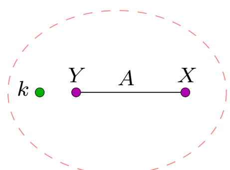
Remark 5.24:
(i) In [34], the double NIM-rep property of the twisted partition functions was proven under the assumption that there is a complete set of defect lines. Now according to formula (5.115), the number of (elementary) defect lines present in a CFT constructed from an algebra object A is given by $\mathrm{tr}\,[Z(A)Z(A)^{\mathrm{t}}]$. This is precisely the number needed for completeness in [34]; thus

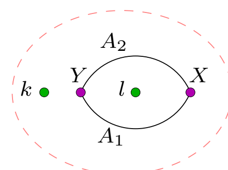
their arguments apply, in agreement with point (vii) of the theorem.

(ii) The properties of $Z_{kl}^{X|Y}$ derived above can already be found explicitly or implicitly in [34,16], with the exception of (iv). This point, together with (v), has a curious interpretation: The twisted torus partition functions of the CFT associated to an algebra A can be expressed in terms of the annulus coefficients of (in general) *different* CFTs possessing the same chiral data – the full CFTs associated to the tensor product algebras $A{\otimes}A^{(-1)}$ and $A^{(1)}{\otimes}A$.

That there exists a relation between defect lines in a CFT and boundary conditions in a product CFT bears some similarity with the 'folding trick' mentioned at the end of section 4.4. However, the folding trick uses boundary states in a CFT of *twice* the central charge of the CFT whose defects are described, whereas the annulus coefficients in point (iv) are those of a CFT with the *same* central charge. The physical interpretation of this observation remains to be clarified.

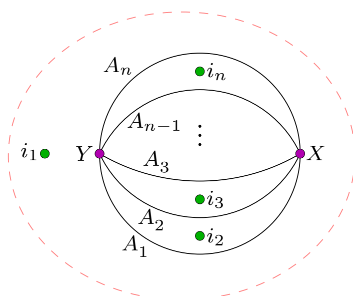
(iii) The structure of a double NIM-rep can in fact be generalised further. So far we have established that annulus coefficients associate a single NIM-rep to left A-modules and that defect lines associate a double NIM-rep to left $A_1{\otimes}A_2$-modules, with suitable algebras A_1 and A_2. We may view the corresponding coefficients schematically as follows:

$\mathrm{A}_{kX}{}^{Y} =$ 

$Z_{kl}^{X|Y} =$ 

(5.166)

What is shown are horizontal sections of ribbon graphs in $S^2{\times}S^1$. More specifically, the figures are cross sections of the graphs (5.119) and (5.151), respectively; vertical ribbons for simple objects and left modules are indicated by filled circles, while the lines symbolise A-ribbons. Analogously, for any pair of left $A_1{\otimes}\cdots{\otimes}A_n$-modules X, Y, the numbers $Z_{i_1,\dots,i_n}^{X|Y}$ defined by

$Z_{i_1,\dots,i_n}^{X|Y} :=$ 

(5.167)

furnish an *n-fold NIM-rep* of the fusion rules, i.e.

$$\sum_R Z_{i_1,\dots,i_n}^{X|R}\, Z_{j_1,\dots,j_n}^{R|Y} = \sum_{k_1,\cdots,k_n\in\mathcal{I}} N_{i_1 j_1}{}^{k_1} \cdots N_{i_n j_n}{}^{k_n}\, Z_{k_1,\dots,k_n}^{X|Y} \,. \tag{5.168}$$

Applications of this structure in string theory remain to be clarified. It is, however, tempting to conjecture that they appear in situations where the world sheet is no longer a smooth manifold. Such world sheets play an important role in the description of string junctions and, more generally, in string networks (see e.g. [131]).

6 Epilogue: Non-commutative geometry in tensor categories

Our results relate rational conformal field theory to the theory of non-commutative algebras and their representations in modular tensor categories. Now a convenient way to think about non-commutative algebras in the tensor category of vector spaces is non-commutative geometry. It is therefore tempting to relate conformal field theory to a version of non-commutative geometry over a modular tensor category.

Consider a compact topological manifold M with measure μ. On the commutative algebra $A = C^0(M)$ of continuous functions a counit $\varepsilon\colon A \to \mathbb{C}$ is provided by the integral, $\varepsilon(f) = \int \mathrm{d}\mu\, f$. This way, A becomes a symmetric Frobenius algebra (though not necessarily a special one). In this sense, the Frobenius algebra A in $\mathcal{C}$ that we used to describe conformal field theory can be regarded as non-commutative measure theory in the tensor category $\mathcal{C}$.

The algebra A itself provides all information needed to analyze correlation functions on arbitrary closed oriented world sheets. The study of boundary conditions and defect lines requires in addition the study of A-modules and A-A-bimodules. They should be thought of as non-commutative vector bundles. As will be discussed in a future publication, our formalism can be extended to unorientable world sheets as well. This requires the choice of a 'conjugation' on A, i.e. the category-theoretic analogue of a conjugation or, in other words, of a *-structure. This can be interpreted as the choice of a real structure in the non-commutative geometry over the tensor category $\mathcal{C}$.

We summarise these ideas in the following table:

world sheet	algebraic data	NC geometry over $\mathcal{C}$
closed oriented	symm. special Frobenius object A	NC measure theory
boundaries	A-modules	NC vector bundles
defect lines	A-A-bimodules	
orientifolds	generalised *-structure on A	real NC geometry
...	...	...

Obviously, in all three columns more entries must be added to arrive at a complete picture. For example, cyclic cohomology, the non-commutative analogue of de Rham cohomology, should play an important role for describing deformations of conformal field theories. Indeed, in the case of complex algebras it is known that the second degree of cyclic cohomology controls deformations of associative algebras with bilinear invariant form (see e.g. [118]). In any case, already on the basis of the presently available evidence it is reasonable to expect that viewing conformal field theory as non-commutative geometry over a tensor category can serve as a fruitful guiding principle in the future.

The emergence of non-commutative structures in conformal field theory does not come as a surprise. There exist *families* of CFTs which are known to yield, in a certain limit, non-commutative field theories [132]. The present results are, however, much closer to a different proposal [133] according to which *every single* conformal field theory gives rise to a non-commutative geometry. These ideas have been formulated within the category of vector spaces. Our results indicate that we can realise them by lifting all relevant structures, algebraic and geometric, to more general tensor categories.

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