Contemporary Mathematics

HU-EP-03/02
hep-th/0301181

Lie algebras, Fuchsian differential equations and CFT correlation functions

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Abstract. Affine Kac-Moody algebras give rise to interesting systems of differential equations, so-called Knizhnik-Zamolodchikov equations. The monodromy properties of their solutions can be encoded in the structure of a modular tensor category on (a subcategory of) the representation category of the affine Lie algebra. We discuss the relation between these solutions and physical correlation functions in two-dimensional conformal field theory. In particular we report on a proof for the existence of the latter on world sheets of arbitrary topology.

1. Some venerable differential equations

One of the surprises in the theory of Kac-Moody algebras is the observation that they give a new and powerful handle on problems that, a priori, do not have an algebraic flavor, including such which are not even genuinely infinite-dimensional. In this contribution we discuss one such application of structures related to affine Kac-Moody algebras: properties of solutions to differential equations.

In the middle of the 18th century Euler, Cauchy and other analysts, the successors of the founding fathers like Newton, Leibniz, and Johann and Jakob Bernoulli, developped a strong interest in "special functions" and series. Arguably, the simplest non-trivial series one can think of is the geometric series

$$f(z) = 1 + z + z^2 + \dots$$

It obeys the ordinary differential equation

$$z\,(1-z)\,f'' + (1-3z)\,f' - f = 0\,.$$

To be of interest in complex analysis, a generalization of the geometric series should, of course, possess a non-zero radius of convergence. Furthermore, a useful guiding principle is that it should also obey a simple differential equation that allows one to determine its main properties. Around the year 1750, such considerations lead

2000 *Mathematics Subject Classification.* 81R10,81T40,33C80,18D10.

Key words and phrases. Hypergeometric functions, conformal blocks, (modular) tensor categories, conformal field theory, WZW theories.

Euler to the *hypergeometric series*

$$F(\alpha,\beta,\gamma;z) := 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}\, z^n \,.$$

It obeys the hypergeometric differential equation

$$z(z-1)\,w'' + [(\alpha+\beta+1)\,z-\gamma]\,w' + \alpha\beta\,w = 0\,. \tag{1.1}$$

This linear differential equation is Fuchsian [**20**], i.e. enjoys the particularly nice property that all three singular points that the coefficient functions acquire upon division by $z(z-1)$ (namely 0, 1, and ∞) are regular singularities. This means that when one approaches such a point in a wedge region of opening angle less than $\pi/2$ the solutions will not diverge stronger than polynomially.

It is quite remarkable that Euler's interest in this series did *not* arise from concrete applications. Indeed, in his famous 1857 paper [**34**] Riemann calls Euler's motivation a "theoretisches Interesse", while he points out there are "numerous applications in physical and astronomic investigations". (This makes a strong point in favour of pure science that is motivated by internal questions.)

Differential equations of the type (1.1) and their solutions are indeed ubiquitous in mathematical physics. They include

- Legendre polynomials, introduced by Legendre around 1800 in his study of gravitational potentials:

$$\frac{1}{\sqrt{1-2\rho z+\rho^2}} = \sum_{n=0}^{\infty} \rho^n P_n(z)\,;$$

- Bessel functions, describing the radial part of the wave function of a free particle in quantum mechanics;
- Laguerre polynomials, the radial part of the hydrogen atom wave functions.

The last two classes of functions are *confluent* hypergeometric functions; they are defined by the limit

$$\Phi(\alpha,\gamma;z) = \lim_{\beta\to\infty} F(\alpha,\beta,\gamma;\tfrac{z}{\beta})\,.$$

So far there is clearly not much algebra in the game. A first hint on the possible relevance of algebra comes from the observation that a deeper understanding of many special functions is afforded by relating them to representation functions of suitable Lie groups. For example, Bessel functions are related to representation functions of irreducible unitary representations of the group of motions of the Euclidean plane. (For a review of the relation between special functions and Lie groups, see e.g. [**26, 8**].)

In this contribution, we consider interesting generalizations of hypergeometric differential equations that have their origin in representation-theoretic structures, involving in particular the representation theory of (classes of) Kac-Moody algebras. They are largely motivated by models of conformal quantum field theory.

In Section 2 we introduce a system of differential equations. In section 3 properties of their solutions are discussed, in particular the asymptotic behaviour and monodromies. These properties possess a convenient interpretation in terms of tensor categories. In section 4 we finally give the underlying motivation from theoretical physics and present a theorem about correlation functions in conformal field theories.

2. The Kniznik-Zamolodchikov equation

Let us select a real compact Lie group G. It admits a bi-invariant metric that allows us to introduce dual bases $\{a_l\}$ and $\{a^l\}$ of the Lie algebra $\mathrm{Lie}\,G$ of G. The element

$$\Omega := \sum_{l=1}^{\dim G} a_l \otimes a^l \ \in \mathrm{Lie}\,G \otimes \mathrm{Lie}\,G$$

does not depend on the choice of bases. Moreover, its action on the tensor product $V_1 \otimes V_2$ of two finite-dimensional G-modules V_1 and V_2 is diagonalizable.

We are interested in functions

$$f: \quad \mathcal{M}_N \to V_1 \otimes \cdots \otimes V_N \tag{2.1}$$

on the moduli space

$$\mathcal{M}_N := \mathbb{C}^{\times N} \setminus \Delta$$

of N mutually distinct points in the complex plane with values in the tensor product of N finite-dimensional G-modules. For such functions we consider the system

$$D_i\, f(z_1, z_2, \dots, z_N) = 0\,, \qquad i = 1, 2, \dots, N\,, \tag{2.2}$$

of differential equations, with differential operators

$$D_i := \kappa\, \partial_{z_i} - \sum_{\substack{j=1\\ i\neq j}}^{N} \frac{\Omega_{ij}}{z_i - z_j}\,. \tag{2.3}$$

Here the symbol Ω_{ij} stands for Ω acting on the tensor product of the ith and jth modules, i.e.

$$\Omega_{ij} = \sum_l \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes R_i(a_l) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes R_j(a^l) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

with the non-trivial factors at the ith and jth positions. Further, κ is a real parameter; for the moment we assume that it is irrational, but eventually, in the context of conformal quantum field theories, we will be particularly interested in certain rational values of κ.

Let us mention one particular physical system in which the differential equation (2.2) arises: a spin chain. Take G to be $SU(2)$ and, for all $i = 1, 2, \dots, N$, V_i the two-dimensional defining representation of $SU(2)$. This system models a one-dimensional metal – a chain of N metal atoms – with each atom having just one electron, with spin described by V_i, in the outermost shell. The tensor product in (2.1) is then the space of physical states. For this spin chain one considers a Hamiltonian

$$H(u) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\Omega_{ij}}{(u - z_i)\,(z_i - z_j)} + \sum_{i=1}^{N} \frac{\mathcal{C}_i}{(u - z_i)^2}$$

depending on parameters $(z_1, z_2, \dots, z_N) \in \mathcal{M}_N$ and $u \in \mathbb{C} \setminus \{z_1, z_2, \dots, z_N\}$. The first term in H describes a hopping from position i to position j, while the second term, involving the quadratic Casimir $\mathcal{C}_i$, is an internal energy that depends on the element of the chain.

One aims at diagonalizing the Hamiltonians $H(u)$ for all values of u simultaneously. One approach to this problem, the so-called Bethe ansatz technique (see e.g. [**25**]), leads to (2.2) as an auxiliary problem [**3, 15**]. Other motivations to study (2.2) comes from low-dimensional quantum field theory, knot theory and, somewhat surprisingly, the theory of modular representations of finite groups of Lie type. These aspects will be addressed later on.

3. Properties of the solutions

We now discuss some properties of the solutions to (2.2). We first study how they behave when one approaches singular points of the coefficients and then have a look at their monodromies. Our discussion in this section follows the exposition in [**5**].

3.1. Asymptotic properties. Two properties of the solutions to (2.2) are immediate:

- By summing over i and using that $\Omega_{ji} = \Omega_{ij}$, one obtains

$$\sum_{i=1}^{N} \partial_{z_i} f(z_1, z_2, \dots, z_N) = 0 .$$

Thus the solutions are invariant under simultaneous translation of all arguments,

$$f(z_1{+}\zeta, z_2{+}\zeta, \dots, z_N{+}\zeta) = f(z_1, z_2, \dots, z_N) .$$

- Second, one finds that

$$\kappa \sum_{i=1}^{N} z_i\, \partial_{z_i} f = \sum_{\substack{i,j=1\\ i<j}}^{N} \Omega_{ij} f .$$

Hence the solutions to (2.2) have a well-defined scaling behaviour:

$$f(\lambda z_1, \lambda z_2, \dots, \lambda z_N) = \lambda^{\kappa^{-1} \sum_{i<j} \Omega_{ij}} f(z_1, z_2, \dots, z_N) .$$

For $N = 3$, these two properties allow us to restrict our attention to the situation that $z_1 = 0$, $z_2 = t$ and $z_3 = 1$. We can therefore simplify the problem and consider functions

$$f : \quad (0,1) \to V_1 \otimes V_2 \otimes V_3$$

that obey the ordinary differential equation

$$\kappa\, f'(t) = \Big(\frac{\Omega_{12}}{t} + \frac{\Omega_{23}}{t-1} \Big) f(t) . \qquad (3.1)$$

This is a Fuchsian differential equation of the type presented in the first section.

Next we study the asymptotics of the solutions $\Gamma(V_1, V_2, V_3)$ to the equations (3.1) on the open interval $(0,1)$ of the real line, a problem analogous to the one studied by Frobenius in [**17**]. Restricting t to be real, we have chosen a well-defined way of approaching the two singular points $t = 0, 1$ of the coefficient functions. Let $v \in V_1 \otimes V_2 \otimes V_3$ be an eigenvector of Ω_{12},

$$\Omega_{12} v = \lambda\, v .$$

If κ is irrational, then there is exactly one solution behaving in the limit $t \to 0$ as

$$t^{\lambda/\kappa} \,(\, v + \text{vector valued analytic function} \,) .$$

Therefore the limit $t \to 0$ provides us with a bijection

$$\Phi_0 : \quad \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3 .$$

One can show that $\Gamma(V_1, V_2, V_3)$ carries the structure of a G-module, and that the bijection Φ_0 is compatible with the G-action and hence constitutes an intertwiner of G-modules. (In particular, Φ_0 restricts to an isomorphism between the respective subspaces of G-invariants.) Similarly, studying the solution close to $t=1$ with the help of eigenvectors of Ω_{23} one obtains a different identification

$$\Phi_1 : \quad \Gamma(V_1, V_2, V_3) \to V_1 \otimes V_2 \otimes V_3 .$$

Combining the two maps we obtain a G-module intertwiner

$$\alpha_{V_1,V_2,V_3} = \Phi_1 \Phi_0^{-1} : \quad (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) .$$

We can now introduce a new non-strict tensor product of G-modules by taking α as the associator that defines the rule for changing brackets in a two-fold tensor product. In short, considering different asymptotics of solutions to differential equations allows us to define a different notion of associativity, and thus a different tensor product, on the category of G-modules. It turns out that this modified tensor product is the relevant one for the conformal field theories in which the differential equations (2.2) arise, and in particular that it fits with the infinite-dimensional algebraic structures that we will encounter in that context below.

One can now proceed to study different limits of solutions to equation (2.2) on the open simplex

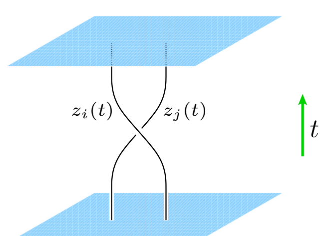
$$z_1 < z_2 < \cdots < z_N$$

to define multiple tensor products. One verifies the consistency of different ways of changing brackets. (Technically, this amounts to showing the pentagon axiom for the associator; for more details we refer to [**5**].)

3.2. Monodromy. A different way to understand properties of the solutions to (2.2) is to analyze the differential operators (2.3). They mutually commute, $[D_i, D_j]=0$, and therefore define a flat connection on a trivial vector bundle over the moduli space $\mathcal{M}_N$. Analytic continuation of the solutions to (2.2) around singular points yields their monodromies. The monodromies generate a representation of the fundamental group $\pi_1(\mathcal{M}_N)$. (Studying the solutions via their monodromies is the point of view of Riemann [**34**].) One can also derive integral representations of the solutions, generalizing those familiar from the hypergeometric function, which provide a tool for calculating the monodromies; see [**36**] and references cited there.

We can use the flat connection to exchange insertion points, as illustrated in the following picture:

(3.2)



Suppose we have selected paths $z_i(t)$ for exchanging the points. Then our task is to integrate the ordinary differential equations

$$\kappa\, f' = \sum_{i<j} \frac{z_i' - z_j'}{z_i - z_j}\, \Omega_{ij} f ,$$

where the prime indicates derivative with respect to t. This gives us a new rule

$$c_{V_1,V_2} := P_{12}\mathrm{e}^{\pi\mathrm{i}\Omega/\kappa} : \quad V_1 \otimes V_2 \to V_2 \otimes V_1$$

for exchanging vectors. This operation does not square to the identity map; as a consequence, what one represents on $V_1 \otimes V_2$ is no longer the permutation group. With some good will, in the picture (3.2) one recognizes braids – and indeed, the fundamental group $\pi_1(\mathcal{M}_N)$ is the pure braid group on N strands.

4. Why is all that interesting?

4.1. Infinite-dimensional structures. To explain the interest of our problem, let us sketch three different applications. The first two will be discussed in the present subsection, the third in subsection 4.4. The algebraic structures to which we have related the analytical problem that we started with have so far been finite-dimensional. But they are actually closely related to infinite-dimensional structures.

(1) The picture (3.2) is reminiscent of knots. Using the fact that the trivial representation is contained in the tensor product of a representation and its dual, one can close the braids and thereby obtain knot invariants and, more generally, invariants of links in three-manifolds. In particular, one obtains Jones' invariant for $G = SU(2)$. This invariant first appeared [**27**] in the study of inclusions of von Neumann algebras. The latter are infinite-dimensional algebras – this is a first indication that infinite-dimensional structures are hidden in the problem under study. As it turns out, most of the analytic structure of von Neumann algebras is indeed not essential for this problem.

The hypergeometric equation discussed above (multiplied with suitable powers of t and $1-t$) is e.g. obtained when considering the invariants in the tensor product of two defining (highest weight $\Lambda_{(1)}$) and two conjugate defining (highest weight $\Lambda_{(n-1)}$) representations of $G = SU(n)$. (Concretely, the relevant hypergeometric functions are ${}_2F_1(\alpha, -\alpha, 1-\beta; t)$ and ${}_2F_1(\alpha+\beta, \alpha-\beta, 1+\beta; t)$ with $\alpha = \frac{1}{n+k}$ and $\beta = \frac{n}{n+k}$, where $k \in \mathbb{Z}_{>0}$ [**29**].) To give another example, for the invariants in the tensor product of four defining representations of $G = SO(n)$ the solutions can be expressed through double contour integrals that are very similar, though different from, the integral representation of the generalized hypergeometric functions ${}_3F_2$, with parameters (appearing as exponents in the integral representation) that are again simple combinations of $\frac{1}{n+k}$ and $\frac{n}{n+k}$ with $k \in \mathbb{Z}_{>0}$ [**18**].

Unfortunately, it seems that this approach to knot theory has not completely lived up to its original hope. Indeed, while the invariants constructed from the monodromies of the solutions can separate knots in a much finer way than classical invariants (they can, for instance, distinguish between a knot and its mirror image), it is still unknown whether they can separate all non-isomorphic knots. Also, to our knowledge these techniques have so far not really lead to a proof of any classical conjecture about knots or the topology of three-manifolds.

(2) The second motivation to study solutions to the equations (2.2) comes from infinite-dimensional algebras which arise in conformal quantum field theories in two dimensions. Such theories have many applications:

▪ In string theory, the notion of a point-particle is replaced by a one-dimensional object, called a string. Instead of a world line, one deals with a two-dimensional world sheet, which is embedded into space-time. After imposing the so-called conformal gauge condition, regarding the space-time coordinates as fields on the world sheet X gives rise to a conformal field theory on X.

- They describe universality classes of two-dimensional critical phenomena. Recently, the theory of critical percolation has received particular attention (for a review, see [**7**]).
- In condensed matter theory, chiral conformal field theory has found applications in the description of (universality classes of) quantum Hall fluids. Conformal field theory on surfaces with boundary has proven to be useful to understand the properties of point-like defects like in the Kondo effect.
- Moreover, the study of conformal field theories has given rise to much interaction between physicists and mathematicians. This interaction has been particularly intense in the theory of infinite-dimensional Lie algebras and, more specifically, of Kac-Moody algebras. But various other mathematical disciplines have been involved as well.

4.2. Correlation functions in quantum field theory. In the setting of point (2) above, a quantum field theory consists, schematically speaking, of two pieces of data:

i) A collection $\mathcal{H}_i$ of spaces of physical states, also called *superselection sectors*. The theory of infinite-dimensional Lie algebras, in particular of affine Kac-Moody algebras, plays an important role in the construction of interesting classes of quantum field theories. In the affine Kac-Moody case, the superselection sectors are obtained from the objects in in the category of integrable modules of finite length and of central charge $\kappa - h(G)$, where $h(G)$ is the dual Coxeter number of G. A toy model that already displays many features of these (sub)categories is the category of finite-dimensional modules of a compact Lie group G.

ii) A collection of *correlation functions*. Let us, for simplicity, consider a quantum field theory defined on the complex plane. Then for every $N \in \mathbb{Z}_{\geq 0}$ the existence of a function on the moduli space $\mathcal{M}_N$ of N distinct points in the plane is required for every N-tuple of states $v_i \in \mathcal{H}_i$. These functions, called (N-point) *correlation functions* or *correlators*, will be denoted by

$$\mathbf{C}_{v_1,\dots,v_N} : \quad \mathcal{M}_N \to \mathbb{C}\,.$$

The physical interpretation of $\mathbf{C}_{v_1,\dots,v_N}$ is that the expression

$$\mathbf{C}_{v_1,\dots,v_N}(z_1, z_2, \dots, z_N) = \langle\, \Phi(v_1; z_1) \cdots \Phi(v_N; z_N)\, \rangle$$

should be regarded as the correlation function of "fields" $\Phi(v_i; \cdot\,)$ that are located at the "insertion points" z_i. Thus a quantum field theory can be thought of as an infinite collection of functions on the moduli spaces $\mathcal{M}_N$.

The symmetries of a quantum field theory are implemented on the correlation functions by differential equations known as *Ward identities*. In the conformal quantum field theories that are formulated with the help of affine Kac-Moody algebras (known as WZW theories), the Ward identities include in particular a system of equations that are of the form of the differential equations (2.2). It is worth mentioning that the correlation functions obtained as solutions to these equations are *exact*, i.e. do not involve any approximation or expansion whose convergence is not under control. In particular there is no need to resort to any kind of perturbation theory – a virtue that results from formulating the theory in algebraic terms.

However, the correlation functions must not only respect the symmetries of the theory, but are in addition subject to several other consistency constraints. For example, the theory must possess so-called cluster properties, i.e.

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda^{\sum_{i=1}^k \Delta_i} \langle\, \Phi(v_1;\lambda z_1) \cdots \Phi(v_k;\lambda z_k)\, \Phi(v_{k+1};z_{k+1}) \cdots \Phi(v_N;z_N)\, \rangle \\ = \langle\, \Phi(v_1;z_1) \cdots \Phi(v_k;z_k)\, \rangle\, \langle\, \Phi(v_{k+1};z_{k+1}) \cdots \Phi(v_N;z_N)\, \rangle \end{aligned} \tag{4.1}$$

for all $1<k<N$, with Δ_i the conformal weight, or scaling dimension, of the field $\Phi(v_i;\cdot\,)$. These constraints express the fact that the interactions of the theory are local in the sense that if we take groups of fields very far apart, their mutual interference can be neglected, so that asymptotically, in the limit of large distances, the correlation function approaches the product of correlators with less insertions.

4.3. Modular tensor categories. To present a third reason for the interest in equation (2.2), a bit more background information is needed. The differential equation (2.2) allows us to endow the category $\mathcal{C}(G)$ of finite-dimensional G-modules with the structure of a braided tensor category. This is a category with a tensor product (typically non-strict, i.e. with a non-trivial associator) and a braiding. A braiding is a collection of prescriptions

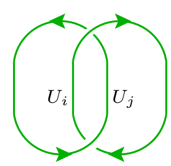
$$c_{V,W}: \quad V\otimes W \to W\otimes V$$

for permuting modules that one obtains from the holonomy with respect to the flat connection (2.3). Moreover, the notion of conjugate modules endows this category with a duality.

One can show – the proof being surprisingly hard [**12, 13, 28**] – that the category obtained this way is equivalent to the category of representations of the quantum group $\mathrm{U}(\mathrm{Lie}\,G)_q$ at a generic value of the deformation parameter q.

Interesting quantum field theories that are realized in two-dimensional physical systems do indeed lead to such categories. Having both braiding and duality morphisms, the categories obtained from these systems also possess a twist; it is related to the conformal weight Δ of the superselection sectors. In the following we restrict our attention to tensor categories which in addition are *modular*, i.e. possess two additional properties:
i) There are only finitely many isomorphism classes of simple objects U_i.
ii) The square matrix $s=(s_{ij})$ that is obtained by taking the trace of twice the braiding of simple objects,

$$s_{ij} \;:=\; \mathrm{tr}\, c_{U_i,U_j} c_{U_j,U_i} \;=$$



is non-degenerate.

Concerning the finiteness of the number of (classes of) simple objects, some comments are in order:

- An example for an algebraic structure with a finite number of irreducible representations are finite groups. Indeed, each finite group gives rise to a modular tensor category, although not in a completely naive way. The reason is, roughly speaking, that Fourier transformation can be defined naively only for functions on abelian groups. There is, however, also a notion of "exotic Fourier transform" [**30**]. These categories appear in so-called orbifold conformal field theories [**9, 6, 10**].

▪ Generic values of the deformation parameter q of the quantum group $\mathrm{U}(\mathrm{Lie}\,G)_q$ correspond to irrational values of the parameter κ that is present in the differential operators (2.3). When κ is chosen to be rational, a truncation to finitely many (isomorphism classes of) simple objects is also possible. (In this case, logarithms can appear in the asymptotics of the solutions to (2.2). These usually do not contribute to the special solutions that in addition satisfy the locality condition (**Loc**) that will be discussed below.) An interesting range of values for κ are the integers larger than the dual Coxeter number $h(G)$ of $\mathrm{Lie}\,G$, which leads to unitary theories. The simple objects of the truncated category then correspond to the level $\kappa-h(G)$ integrable irreducible highest weight modules of the untwisted affine Kac-Moody algebra based on $\mathrm{Lie}\,G$. For the representation category of the quantum group $\mathrm{U}_q(\mathrm{Lie}\,G)$, an analogous truncation is possible when q is a root of unity; this truncation is closely connected to the truncation in the Lie algebra case, with the relation between the deformation parameter and κ given by $q=\mathrm{e}^{2\pi\mathrm{i}/\kappa}$.

4.4. Modular representations. Now we are in a position to present the third motivation for the study of the differential equations (2.2): modular representations of finite groups like $SL(\mathbb{F}_p)$ (with $\mathbb{F}_p$ the field with p elements) on vector spaces over fields of the same characteristic p. It is amazing that such a purely algebraic problem requires (complex-) analytic tools as well as structures arising in connection with infinite-dimensional Lie algebras. But the study of such representations is difficult indeed, and even their dimension was unknown ten years ago.

The central idea is that in finite characteristic a similar truncation happens as upon deformation. One therefore relates representation theoretic problems – most importantly, the problem of describing the composition series of the highest weight module obtained via the Borel-Bott-Weil construction – to problems for quantum groups. Their representation category, or rather a certain truncation thereof, carries the structure of a modular tensor category. One shows that it is (anti-)equivalent to a category of modules over an affine Lie algebra which obtains its structure of a modular tensor category precisely from (2.2). In the latter category, it is comparatively easy to work with character formulae and related structures. (For a review, see [**2**] and the last section of [**31**].)

For completeness, we finally mention that to some extent the logic we presented can be reversed. That is, each modular category gives rise to monodromy data, in fact for arbitrarily many insertion points on a Riemann surface of any genus. However, one and the same modular tensor category (and hence monodromy data) can appear as the representation category of more than one conformal field theory. Indeed one even knows of examples where infinitely many conformal field theories possess the same modular tensor category: those based on the groups $SO(n)$ and level 1, with fixed value of n mod 16.

5. Existence of correlation functions

5.1. Complexification. The schematic summary of (two-dimensional conformal) quantum field theories presented above is, of course, an oversimplification. In particular, their structure is more complicated in the following respect. The world sheet – the two-dimensional manifold X on which the theory is living – need not be the complex plane, or equivalently the Riemann sphere, but can be a Riemann surface of arbitrary genus. It may also have a boundary, as happens for the world sheet of open strings, and need not be orientable. Furthermore, to obtain well-understood differential equations such as (2.2), one does not directly study the correlation functions on the world sheet X that arises in the physical application, but (multi-valued) functions on a different two-dimensional manifold.

This situation is familar from potential theory. Harmonic functions on $\mathbb{R}^2$ are either analytic in $z = x + \mathrm{i}y$ or analytic in $\bar{z} = x - \mathrm{i}y$. A convenient trick when working with harmonic functions is to consider instead of functions on $\mathbb{R}^2$ functions on a complex z-plane and on a complex $\bar{z}$-plane. This allows us to work in a complex setting. Note that that to achieve this we must work with a *double* of the manifold.

This manipulation becomes more transparent when one considers the case that X has a boundary. Then the prescription, including again a doubling of the space, is well known from classical electrodynamics. Consider a point charge at position $y \in \mathbb{R}^n$ in front of an infinite grounded hyperplane located at $x^n = 0$. Without boundary, the electrostatic potential obeys the Poisson equation

$$-\Delta_x V_y(x) = q\,\delta(x-y)\,,$$

which is solved by the Greens function; in two dimensions,

$$V_y(x) = G(x,y) = -\tfrac{1}{2\pi}\,\log\,|x-y|\,.$$

To include the effects of the grounded plane, one introduces a mirror charge − the position of which is given by reflection σ about the plane. The potential in the presence of the plane then reads

$$V(x) = G(x,y) - G(x,\sigma y)$$

Notice that again the space we use for solving the problem is a double cover of the original space on which the physical problem is formulated.

Further, in two-dimensional conformal field theory it is natural [**35**] to endow the space X with the structure of a real scheme. The simple differential equations (2.2) – the Ward identities – and their solutions – often called *conformal blocks* – then live on the complexification $\hat{X}$ of X, which is a complex scheme. Let us give some examples. If X does not have any real point (intuitively, in the language of manifolds: if the boundary ∂X is empty), then, as a manifold, $\hat{X}$ is just the total space of the orientation bundle over X. In particular, when X is orientable and $\partial X = \emptyset$, then the double $\hat{X}$ is the disjoint union of two copies of X with opposite orientation, $\hat{X} = X \sqcup (-X)$. Two more examples: the complex double of the disk is the Riemann sphere, and the double of an annulus is a torus. The double $\hat{X}$ carries the action of an anticonformal involution σ, which is nothing but the action of the Galois group $\mathcal{G}al(\mathbb{C},\mathbb{R}) \cong \mathbb{Z}_2$ on the complexification. One may identify X with the quotient of $\hat{X}$ by this action,

$$X = \hat{X}/\{1,\sigma\}\,.$$

The insertion points on $\hat{X}$ are the pre-images, under the natural projection from $\hat{X}$ to X, of the insertion points p_i on X. Thus except when $p \in \partial X$, a field insertion $\Phi(v;p)$ on X gives rise to two insertions on $\hat{X}$.

5.2. Consistency conditions. We have thus arrived at what is known as the *principle of holomorphic factorization*: the physical correlation functions on X are specific solutions to the differential equations (2.2) on the double $\hat{X}$. These specific solutions must obey the following additional consistency constraints:

(**Loc**) They must be *local*, i.e. genuine functions on X; thus their monodromies must be trivial. This means that one looks for solutions to (2.2) on $\hat{X}$ that are invariant under the subgroup of the mapping class group of $\hat{X}$ that commutes with σ. This subgroup can be identified with the mapping class group of X. This requirement is known as modular invariance; it includes also moduli of the conformal structure on X.

(**Fac**) They must possess *factorization* properties, which generalize the cluster property (4.1).

(For a precise statement of factorization see, for instance, theorems 3.9 and 3.10 of [**16**].)

A central problem for understanding conformal quantum field theories is to select among the conformal blocks on the double $\hat{X}$ – solutions to differential equations with non-trivial monodromies – a consistent collection of correlation functions. A priori it is far from clear whether this is possible at all. On the other hand, for specific classes of theories necessary conditions on correlation functions have been discussed already long ago. In particular, consequences of the locality condition (**Loc**) for correlation functions on surfaces of genus 0 have been discussed in the form of associativity constraints on the operator product expansion, see e.g. [**11, 29**], and in the form of modular invariance constraints on the partition function (i.e., 0-point correlation function) on the torus (genus 1), the latter possessing always the so-called C-diagonal solution. However, at the time it was not known whether these solutions are part of a system of correlation functions on world sheets of arbitrary genus that solves *all* the constraints.

Only recently, a complete affirmative answer to the existence question could be given in the so-called Cardy case: It was shown [**16**] that indeed the C-diagonal torus partition function is part of a system of correlators satisfying all constraints. The proof is constructive, describing every correlator as the invariant of a concrete ribbon graph in a three-manifold. This description is possible due to the fact that the representation category of the CFT – in the present paper a full subcategory of the representation category of an infinite-dimensional Lie algebra – possesses the structure of a modular tensor category.

What is not settled by the result of [**16**] is the question of uniqueness of the solution. For the case of the torus partition function and for some very specific correlators at genus zero, solutions to (**Loc**) that are not of the C-diagonal form were also already known for quite a while. In the latter examples the check of locality involves application of rather special identities. For instance, for certain 4-point correlators in WZW theories based on the group $SU(n)\times SU(n')$, for which the blocks are products of blocks for $SU(n)$ and for $SU(n')$, one finds non-C-diagonal local combinations at the special values $k=n'$ and $k'=n$ of the levels, using e.g. [**19**] a relation between products of hypergeometric functions and generalized hypergeometric functions as well as ${}_3F_2(\alpha,-\alpha,\frac{1}{2};\beta,1-\beta;t)+\frac{t}{2nn'}\,{}_4F_3(1+\alpha,1-\alpha,\frac{3}{2},1;1+\beta,2-\beta,2;t)=1$. (Compare also appendix E of [**1**] for an application to the Kondo effect.)

5.3. The main theorem. To find a collection of correlation functions that obey the two constraints (**Loc**) and (**Fac**) is a highly non-trivial problem. Two strategies are a priori envisageable: the first one aims at reducing these constraints to a small set of necessary and sufficient conditions. Due to the complexity of the problem, this approach has quite a few pitfalls (see e.g. [**4**]). In fact, the most popular way-out has been to work with a set of necessary conditions and to hope that they are sufficient as well. Here we adopt an opposite strategy and give a simple *sufficient* condition that guarantees the existence of a consistent system of correlation functions for a conformal field theory. Our approach is based on a combination of the structure of braided tensor categories with concepts from non-commutative algebra. It leads to the following central result:

> **Theorem** [**21**]:
> When the monodromies of a system of conformal blocks are described by a modular tensor category, then a consistent set of correlation functions solving (**Loc**) and (**Fac**) exists for each quadruple $(A, m, \eta, \varepsilon)$ of the following data: A is an object in the modular tensor category (and thus a superselection sector), m is a multiplication
>
> $$m: \quad A \otimes A \to A$$
>
> that is associative with respect to the associator,
>
> (5.1) $$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \circ \alpha_{AAA} \,,$$
>
> and $\eta \in \mathrm{Hom}(\mathbf{1}, A)$ a unit morphism with respect to m.
> Finally, the morphism $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ is an appropriate generalization of a linear form that turns the algebra (A, m, η) into a so-called *symmetric special Frobenius algebra*.

(For precise definitions we refer to [**22**].)

The proof of this statement has not yet been fully published. It uses heavily non-commutative algebra in tensor categories. Here we can only make a few short comments:

1) The only non-linear equation that has to be solved for turning the object A into a symmetric special Frobenius algebra is the associativity requirement (5.1).

2) The proof makes use of a three-manifold M_X that is bounded by $\hat{X}$ and that has X as a retract. Every correlation function can be described as the invariant of a ribbon graph in M_X, with the building blocks of the graph corresponding to morphisms in the modular tensor category. (An important ingredient in the construction of the graph is a triangulation of X [**21**].) The consistency constraints (**Loc**) and (**Fac**) can then be verified by manipulating ribbon graphs in M_X.

3) For various physical reasons one is interested in manifolds X with non-empty boundary. Correlation functions on such X depend on the choice of a "boundary condition" on ∂X. The treatment of this situation is possible owing to a simple description of boundary conditions: they correspond to modules (in the tensor category $\mathcal{C}$) of the Frobenius algebra A. This allows us to apply representation theoretic methods [**24, 33**] to the study of boundary conditions in conformal field theory, and thus e.g. to describe D-branes in certain string theoretic models on Calabi-Yau spaces. In a similar way, so-called "defect lines" correspond to A-A-bimodules.

4) It is not known so far (though we expect it to be the case) whether the existence of an associated quadruple $(A, m, \eta, \varepsilon)$ is not only a sufficient, but also a necessary condition for a solution to the constraints (**Loc**) and (**Fac**) to exist, and in particular, whether *every* solution is obtained in the way described by the theorem. In any case, solutions and quadruples $(A, m, \eta, \varepsilon)$ are *not* in bijection. Rather, Morita equivalent algebras give identical solutions. When one combines this insight with

orbifold techniques, it allows one to give a rigorous proof of T-dualities and algebraic versions of mirror symmetry.

5) The construction of [**21**] can be generalized in such a way that correlation functions can also be assigned to manifolds without orientation [**23**]. In this case, the datum of a symmetric special Frobenius algebra is not enough to determine the theory. Rather, one must in addition require that the algebra (A, m, η) is isomorphic to the opposite algebra $(A, m \circ c_{A,A}, \eta)$. The choice of an algebra isomorphism, if it exists, enters as an additional datum.

6. Conclusions

Our starting point was the analytic problem of selecting consistent correlation functions in a conformal field theory. We have then realized that an algebraization of this problem allows us to obtain general proofs of consistency. Incidentally, we also gain considerable computational power. In particular, one can express what physicists call the "structure constants of the operator product expansion" in terms of invariants of links in three-manifolds, which makes them explicitly computable in concrete models and allows one to prove some of their properties. Another virtue of this approach is that it reduces some long-standing physical questions to standard problems in algebra and representation theory. Here are some examples:

- As already discussed, we expect that the classification of CFTs with given chiral data $\mathcal{C}$ amounts to classifying Morita classes of symmetric special Frobenius algebras in the category $\mathcal{C}$. In particular, physical modular invariant partition functions of so-called [**32**] automorphism type are classified by the Brauer group of $\mathcal{C}$.
- The classification of boundary conditions and defect lines (that respect the underlying chiral symmetry) is reduced to the standard representation theoretic problem of classifying modules and bimodules. As a consequence, powerful methods like induced modules and reciprocity theorems are at our disposal.
- The problem of deforming CFTs is related to the problem of deforming algebras; this problem is controlled by a suitable cohomology theory. For the moment, the only known results in this direction are rigidity theorems [**14**]: a rational CFT cannot be deformed within the class of rational CFTs. This is but one, though in our opinion not the least important, reason to make an effort to gain a better understanding of non-rational conformal field theory.

Indeed, in our opinion the biggest challenge in the theory is to go beyond the rational case, i.e. admit categories that are not semi-simple any more and which do not have such benign duality properties. Not too much is known about such theories on a rigorous level. This is partly due to the lack of a class of well-understood examples. In the rational case, the theory of Kac-Moody algebras – or rather, quite specifically, of affine Lie algebras – has been of enormous help for the development of conformal field theories. Conversely, the lack of knowledge concerning non-rational conformal field theories can be regarded as reflecting our comparatively poor understanding of the representation theory of non-compact forms of Kac-Moody algebras of affine type.

Acknowledgments:
We are grateful to Jürg Fröhlich for helpful comments on the manuscript.

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