Solutions for exercise sheet # 12
Topics in representation theory WS 2017

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Exercise 50

1. As the product is compatible with the grading, so is the super commutator. Graded skew symmetry is clear. For the graded Jacobi identity simply write out all terms

$$(-1)^{|a||c|}\big(abc - (-1)^{|b||c|}acb - (-1)^{|a|(|b|+|c|)}bca + (-1)^{|b||c|+|a|(|b|+|c|)}cba + \text{cycl.}$$
$$= (-1)^{|a||c|}abc - (-1)^{|b||a|}bca - (-1)^{(|a|+|b|)|c|}acb + (-1)^{(|c|+|a|)|b|}cba + \text{cycl.}$$

The first two summands will cancel in the cyclic sum, as will the last two.

2. For $V = V_0 \oplus V_1$ write an element of $gl(V)$ in block form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A : V_0 \to V_0$, $B : V_1 \to V_0$, etc. We define $\phi : gl(V) \to gl(\Pi V)$ as

$$\phi \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} D & C \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix}$$

Note that ϕ is an even linear map. Using that ϕ is just conjugation with a matrix, it is easy to check that for $M, N \in gl(V)$ homogeneous we have

$$\phi([M, N]) = \phi(M)\phi(N) - (-1)^{|M||N|}\phi(N)\phi(M) = [\phi(M), \phi(N)] \ .$$

Exercise 51

(Sketch only). As the Jacobi identity is cyclically invariant, there are just 4 cases to check, which we label by the parity of the three elements: (0,0,0), (0,0,1), (0,1,1), (1,1,1).

- (0,0,0): holds by assumption, as L_0 is a Lie algebra.
- (0,0,1): After reshuffling terms, this is the statement that L_1 is an L_0-module.
- (0,1,1): After reshuffling terms, this is the statement that β is an L_0-intertwiner.

- (1,1,1): Let $c = rx + sy + tz$ for $r, s, t \in k$ and $x, y, z \in L_1$. Abbreviate, for $u, v, w \in L_1$, $B(u, v, w) = [u, [v, w]] = -\beta(v, w).u$. Note that $B(u, v, w) = B(u, w, v)$ (as β is symmetric) By assumption, $B(c, c, c) = 0$. By multilinearity of B, this becomes

  $$0 = B(c, c, c) = r^3 B(x, x, x) + r^2 s B(x, x, y) + r^2 t B(x, x, z) + \dots \ ,$$

  with 9 terms altogether. Since $char(k) = 0$, the coefficient of these monomials must all be independently zero. Collecting all terms multiplying rst shows

  $$\begin{aligned} 0 &= B(x, y, z) + B(x, z, y) + B(y, x, z) + B(y, z, x) + B(z, x, y) + B(z, y, x) \\ &= 2B(x, y, z) + 2B(y, z, x) + 2B(z, x, y) \ , \end{aligned}$$

  where in the second step we used symmetry of B in the last two arguments. This is the graded Jacobi identity for 3 odd elements.

Exercise 52

To show that $sl(m|n)$ is a Lie-super subalgebra of $gl(m|n)$, we must show that the super-trace of the graded commutator is zero, provided the super-trace of its two arguments is zero. In fact this holds already for two arbitrary elements of $gl(m|n)$, as we now check (and so $sl(m|n)$ is even an ideal in $gl(m|n)$, not just a subalgebra).
Let $A, B \in gl(m|n)$ be homogeneous elements. We need to show

$$str(AB - (-1)^{|A||B|} BA) = 0$$

First note that the super-trace of any odd endomorphism of $k^{m|n}$ is zero by definition. Thus the only non-trivial cases are $|A| = |B| = 0$ and $|A| = |B| = 1$

- $|A| = |B| = 0$: Write A_{00}, A_{11}, B_{00}, B_{11} for the non-zero components. Then

  $$str(AB - (-1)^{|A||B|} BA) = tr_{k^{m|0}}(A_{00}B_{00} - B_{00}A_{00}) - tr_{k^{0|n}}(A_{11}B_{11} - B_{11}A_{11}) \ .$$

  Here, each trace is zero separately by cyclicity of the trace.

- $|A| = |B| = 1$: Write A_{01}, A_{10}, B_{01}, B_{10} for the non-zero components. Then

  $$\begin{aligned} &str(AB - (-1)^{|A||B|} BA) \\ &= tr_{k^{m|0}}(A_{01}B_{10} + B_{01}A_{10}) - tr_{k^{0|n}}(A_{10}B_{01} + B_{10}A_{01}) \\ &= tr_{k^{m|0}}(A_{01}B_{10} + B_{01}A_{10}) - tr_{k^{m|0}}(B_{01}A_{10} + A_{01}B_{10}) \ , \end{aligned}$$

  where we applied cyclicity only to the second summand. Now the two traces cancel each other.

If we had used the ordinary trace instead of the super trace, we would in general not obtain a subalgebra, as the ordinary trace of a supercommutator may be nonzero. In the above arguemnt, the case $|A| = |B| = 1$ would fail as there would be a "+" between the two traces.

Exercise 53

We need to check that $[\varphi(A), \varphi(B)] = \varphi([A, B])$. Abbreviate $\tilde{A}_{ij} = -(A\eta)_{ij}$. Then

$$[\varphi(A), \varphi(B)] = \tfrac{1}{16} \sum_{i,j,k,l} \big(\tilde{A}_{ij} \tilde{B}_{kl} e_i e_j e_k e_l - \tilde{B}_{kl} \tilde{A}_{ij} e_k e_l e_i e_j \big) = (*) \ .$$

Now one checks that

$$e_l e_i e_j = e_i e_j e_l - 2\eta_{li} e_j + 2\eta_{lj} e_i$$

and

$$e_k e_l e_i e_j = e_i e_j e_k e_l - 2\eta_{ki} e_j e_l + 2\eta_{kj} e_i e_l - 2\eta_{li} e_k e_j + 2\eta_{lj} e_k e_i \ .$$

Inserting this gives

$$(*) = \tfrac{1}{8} \sum_{r,s} \big((\tilde{A}^t \eta \tilde{B})_{kl} - (\tilde{A} \eta \tilde{B})_{kl} + (\tilde{B} \eta \tilde{A})_{kl} - (\tilde{B}^t \eta \tilde{A})_{kl} \big) e_k e_l$$

Using $\tilde{A}^t = -\tilde{A}$ and $\eta^2 = id$ one see that the rhs is equal to $\frac{1}{4} \sum_{k,l} ((-AB + BA)\eta)_{kl} e_k e_l = \varphi([A, B])$.

Exercise 54

The Clifford algebra is filtered by $0 = F_0 \subset F_1 \subset F_2 \subset \cdots$, where F_n is just the image of $T^{\leq n}(V)$ under the projection $T(V) \to Cl(V, q)$. The exterior algebra is filtered in the same way, we write $\Lambda^{\leq n} V$ for the filtered components. By construction, s maps $\Lambda^{\leq n} V$ to F_n.
We show by induction on n that s is surjective. For $n = 1$ this is clear. Suppose $s : \Lambda^{\leq n} V \to F_n$ is surjective. Now F_{n+1} is spanned by F_n and elements $v_1 \cdot v_2 \cdots v_{n+1}$. Since we can freely reorder factors in F_{n+1} up to signs when exchanging two factors and up to terms in F_n (in fact: F_{n-1}), we have

$$s(v_1 \wedge \cdots \wedge v_{n+1}) = v_1 \cdot v_2 \cdots v_{n+1} + \omega \quad , \quad \omega \in F_n \ .$$

By induction hypothesis, there exists $\nu \in \Lambda^{\leq n}$ such that $s(\nu) = \omega$. Thus $v_1 \cdot v_2 \cdots v_{n+1} = s(v_1 \wedge \cdots \wedge v_{n+1} + \nu)$.

If V is finite-dimensional, we are done: For $n = \dim(V)$ we see that $s : \Lambda(V) \to Cl(V, q)$ is surjective, hence bijective. This is really how I meant the problem, but I forgot to say 'finite-dimensional', so let's look at the infinite dimensional case.

Surjectivity: Let $x \in Cl(V, q)$. Then there is an n such that $x \in F_n$. The above argument then applies showing surjectivity.

Injectivity: Let $U \subset V$ be a finite-dimensional subspace of V. We have seen above that $s|_{\Lambda(U)}$ is injective (as it is a bijection to $Cl(U, q|_U)$).
It remains to show that every $y \in \Lambda(V)$ is contained in $\Lambda(U)$ for some finite-dimensional $U \subset V$. This can be seen as follows: y can be written as a finite sum of terms of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ (where m and the v_i vary from summand to summand). Consider the span $U \subset V$ of all the (finitely many) vectors of V occurring in this expression. By construction, $y \in \Lambda(U)$.