Solutions for exercise sheet # 11
Topics in representation theory WS 2017

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Exercise 45

We will show that $-1 \notin Spin^0_{1,1}$. By 3.3, Lem. 1 we have $Cl_{1,1} \cong \mathrm{Mat}(2,\mathbb{R})$. A possible isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then $v = \alpha e_1 + \beta e_2 \in Cl_{1,1}$ such that $q(v) = \pm 1$ are mapped to $M = \begin{pmatrix} \beta & \alpha \\ -\alpha & -\beta \end{pmatrix}$ such that $\det(M) = \pm 1$ (this uses that $q(v) = \alpha^2 - \beta^2 = \det(M)$). The group generated by such M is contained in matrices of the form

$$\Big\{ M = \begin{pmatrix} \beta & \alpha \\ -\alpha & -\beta \end{pmatrix} \Big| \det(M) = \pm 1 \Big\} \ \cup \ \Big\{ N = \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix} \Big| \det(N) = \pm 1 \Big\} .$$

Both id and $-id$ are contained in the second component and have determinant 1. Hence the connecting path must have $\det N = 1$ along the path. But then $\gamma^2 - \delta^2 = 1$, that is, $\gamma^2 = 1 + \delta^2 \geq 1$. Any path connecting id to $-id$ would need to include a point where $\gamma = 0$, which is impossible.

Exercise 46

Uniqueness: Relation 2, applied to $a = y \in U'$ and $b = 1$ implies that $x \lrcorner 1 = 0$. One can now use Relation 2 to define $x \lrcorner (-)$ inductively on $\Lambda^r(U')$.

Existence: The tedious step is to verify relation 2. We reduce the problem to the case $r = 1$.

Claim: Suppose a given map $x \lrcorner (-)$ satisfies relation 1 and relation 2 in the case $r = 1$. Then it satisfies relation 2 for all r.
Proof: By induction. Suppose relation 2 holds for some $r - 1 \geq 1$. Let now $a \in \Lambda^r(U')$. All such a are linear combinations of $a = a' \wedge a''$, where $a' \in U'$ and $a'' \in \Lambda^{r-1}(U')$. Thus we may as well assume that $a = a' \wedge a''$ (relation 2 is bilinear). Then

$$\begin{aligned}
x \lrcorner (a \wedge b) &= x \lrcorner (a' \wedge a'' \wedge b) \overset{(*)}{=} (x \lrcorner a') \wedge a'' \wedge b - a' \wedge x \lrcorner (a'' \wedge b) \\
&\overset{(**)}{=} (x \lrcorner a') \wedge a'' \wedge b - a' \wedge \Big((x \lrcorner a'') \wedge b + (-1)^{r-1} a'' \wedge (x \lrcorner b) \Big) \\
&= \Big((x \lrcorner a') \wedge a'' - a' \wedge (x \lrcorner a'') \Big) \wedge b + (-1)^r a' \wedge a'' \wedge (x \lrcorner b) \\
&\overset{(*)}{=} x \lrcorner (a' \wedge a'' \wedge b) + (-1)^r a' \wedge a'' \wedge (x \lrcorner b) \\
&= (x \lrcorner a) \wedge b + (-1)^r a \wedge (x \lrcorner b) \ ,
\end{aligned}$$

where (*) is the assumption that relation 2 holds for $r = 1$ and (**) is the induction hypothesis for $r - 1$. This completes the proof of the claim.

To give a concrete linear map $x \lrcorner (-)$ we work in a basis. Let $e_1, \dots, e_n$ be a basis of U'. Write $\alpha_i := \beta(x, e_i) \in \mathbb{C}$. From Prop. 3.1.9 we know that $e_1^{\delta_1} \wedge \dots \wedge e_n^{\delta_n}$ is a basis of $\Lambda(U')$. Define $x \lrcorner (-)$ on this basis as

$$x \lrcorner (e_1^{\delta_1} \wedge \dots \wedge e_n^{\delta_n}) := \sum_{k=1}^{n} \delta_k (-1)^{\sum_{l=1}^{k-1} \delta_l} \alpha_k \cdot e_1^{\delta_1} \wedge \cdots e_k^{\delta_k - 1} \cdots \wedge e_n^{\delta_n}$$

Relation 1 now follows by considering all basis vectors where only one δ_k is non-zero.
By the above claim, we only need to check relation 2 for $r = 1$ (and, by bilinearity, only on a basis). We will check it on

$$a = e_t \quad , \quad b = e_1^{\delta_1} \wedge \dots \wedge e_n^{\delta_n} \ .$$

We distinguish two cases: $\delta_t = 0$ and $\delta_t = 1$. Let us start with $\delta_t = 1$. Then $a \wedge b = 0$, and it remains to check that the rhs of relation 2 is zero, too. We compute

$$\begin{aligned}
&(x \lrcorner e_t) \wedge b - e_t \wedge (x \lrcorner b) \\
&= \alpha_t b - e_t \wedge \Big(\sum_{k=1}^{n} \delta_k (-1)^{\sum_{l=1}^{k-1} \delta_l} \alpha_k \cdot e_1^{\delta_1} \wedge \cdots e_k^{\delta_k - 1} \cdots \wedge e_n^{\delta_n} \Big) \\
&\overset{(*)}{=} \alpha_t b - e_t \wedge \Big((-1)^{\sum_{l=1}^{t-1} \delta_l} \alpha_t \cdot e_1^{\delta_1} \wedge \cdots e_t^{0} \cdots \wedge e_n^{\delta_n} \Big) \\
&= \alpha_t b - \alpha_t \cdot e_1^{\delta_1} \wedge \cdots e_t^{1} \cdots \wedge e_n^{\delta_n} = 0 \ ,
\end{aligned}$$

where in (*) we used that only the term for $k = t$ will give a non-zero contribution (all giving zero because of the leading $e_t \wedge (-)$).
Next consider the case $\delta_t = 0$. Let $\hat{\delta}_k = \delta_k$ for $k \neq t$ and $\hat{\delta}_t = 1$. Let

$$c = e_1^{\hat{\delta}_1} \wedge \dots \wedge e_n^{\hat{\delta}_n}$$

The lhs of relation 2 is

$$(-1)^{\sum_{l=1}^{t-1} \delta_l} x \lrcorner c \ ,$$

where $x \wedge c$ abbreviates the definition of $x \lrcorner (-)$ on the basis. On the rhs we compute

$$
\begin{aligned}
&(x\lrcorner e_t)\wedge b - e_t\wedge (x\lrcorner b)\\
&= \alpha_t b - e_t \wedge \Big(\sum_{k=1,k\neq t}^{n} \delta_k (-1)^{\sum_{l=1}^{k-1}\delta_l}\alpha_k \cdot e_1^{\delta_1}\wedge\cdots e_k^{\delta_k-1}\cdots\wedge e_n^{\delta_n}\Big)\\
&= \alpha_t b - \sum_{k=1,k<t}^{n} \delta_k(-1)^{\sum_{l=1}^{k-1}\delta_l}\alpha_k(-1)^{-1+\sum_{l=1}^{t-1}\delta_l}\cdot e_1^{\hat\delta_1}\wedge\cdots e_k^{\hat\delta_k-1}\cdots\wedge e_n^{\hat\delta_n}\\
&\qquad - (-1)^{\sum_{l=1}^{t-1}\delta_l}\sum_{k=1,k>t}^{n}\delta_k(-1)^{\sum_{l=1}^{k-1}\delta_l}\alpha_k\cdot e_1^{\hat\delta_1}\wedge\cdots e_k^{\hat\delta_k-1}\cdots\wedge e_n^{\hat\delta_n}\\
&= (-1)^{\sum_{l=1}^{t-1}\delta_l}\Big(\alpha_t\hat\delta_t(-1)^{\sum_{l=1}^{t-1}\hat\delta_l}\alpha_k\cdot e_1^{\hat\delta_1}\wedge\cdots e_t^{\hat\delta_t-1}\cdots\wedge e_n^{\hat\delta_n}\\
&\qquad + \sum_{k=1,k<t}^{n}\hat\delta_k(-1)^{\sum_{l=1}^{k-1}\hat\delta_l}\alpha_k\cdot e_1^{\hat\delta_1}\wedge\cdots e_k^{\hat\delta_k-1}\cdots\wedge e_n^{\hat\delta_n}\\
&\qquad + \sum_{k=1,k>t}^{n}\delta_k(-1)^{\sum_{l=1}^{k-1}\hat\delta_l}\alpha_k\cdot e_1^{\hat\delta_1}\wedge\cdots e_k^{\hat\delta_k-1}\cdots\wedge e_n^{\hat\delta_n}\Big)\ ,
\end{aligned}
$$

which equals the lhs.

Exercise 47

Let $x \in U$ be arbitrary. We will show inductively that $x\lrcorner(x\lrcorner c) = 0$ for all $c \in \Lambda(U')$. For $c = 1$ and $c \in U'$ that is clear. Assume now the claim is verified on $\Lambda^s(U')$ for $s \leq r-1$. For $c \in \Lambda^r(U')$, it is enough to show the claim in elements of the form $a \wedge b$, where a, b have degree strictly less than r. We get, using relation 2 (and omitting $\lrcorner$ for brevity):

$$
\begin{aligned}
x(x(a\wedge b)) &= x\big((xa)\wedge b + (-1)^r a\wedge (xb)\big)\\
&= (xxa)\wedge b + (-1)^{r+1}(xa)\wedge(xb) + (-1)^r(xa)\wedge(xb) + (-1)^{r+r}a\wedge(xxb)\ ,
\end{aligned}
$$

which, together with the induction hypothesis, gives zero.
Let $x = u + u'$ be a general element of $U \oplus U'$ (with $u \in U$, $u' \in U'$). Using the above, we compute, for any $y \in \Lambda(U')$,

$$
\begin{aligned}
x^2 y &= u\lrcorner(u\lrcorner y) + u\lrcorner(u'\wedge y) + u'\wedge(u\lrcorner y) + u'\wedge(u'\wedge y)\\
&= u\lrcorner(u'\wedge y) + u'\wedge(u\lrcorner y) = (u\lrcorner u')\wedge y\\
&= \beta(u,u')y\ .
\end{aligned}
$$

On the other hand, $q(x) = \frac{1}{2}\beta(x,x) = \beta(u,u')$. Thus the relation in the universal property holds, giving the map *rho* as required.

Exercise 48

2. $\Rightarrow$ 1.: Set $\varphi(\sum_k z_k w_k) := \sum_k \bar{z}_k w_k$. The required properties of φ are then immediate.

1. $\Rightarrow$ 2.: Since $\varphi^2 = id$, φ is diagonalisable (over $\mathbb{R}$) and has eigenvalues ± 1 (use the images of the idempotents $\frac{1}{2}(id \pm \varphi)$ to see this). Let $W = W_+ \oplus W_-$ the decomposition of W, seen as a real vector space, into the (real) eigenspaces $W_\pm$ of φ. Note that if $x \in W_+$, then $ix \in W_-$, and vice versa, so that $i \cdot (-)$ is an $\mathbb{R}$-linear isomorphism $W_+ \to W_-$. Let $w_1, \dots, w_n$ be an $\mathbb{R}$-basis of W_+.

Claim: $w_1, \dots, w_n$ is a $\mathbb{C}$-basis of W.

Proof: Since the iw_k span W_- over $\mathbb{R}$, it is clear that the w_k span W over $\mathbb{C}$. For linear independence, consider separately the components of $\sum_k (a_k + ib_k) w_k = 0$ in $W_\pm$.

Claim: The matrix entries of $\rho(a)$ are real.

Proof: By assumption, $\varphi(\rho(a)x) = \rho(a)\varphi(x)$ for all $a \in A$, $x \in W$. For $x = w_k$, we have $\varphi(w_k) = w_k$ and this condition reads

$$\varphi(\rho(a)w_k) = \rho(a)w_k \ .$$

Writing this out in matrix elements gives

$$\varphi(\sum_j \rho(a)_{jk} w_j) = \sum_j \rho(a)_{jk} w_j \ .$$

By antilinearity, the lhs is equal to $\sum_j \overline{\rho(a)_{jk}} w_j$, which shows that $\overline{\rho(a)_{jk}} = \rho(a)_{jk}$.

Exercise 49

1. As A is semisimple over $\mathbb{R}$, $A' := \mathbb{C} \otimes_{\mathbb{R}} A$ is semisimple over $\mathbb{C}$ (use e.g. that A is a direct sum of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, each of which produces one or more matrix algebras over $\mathbb{C}$ after tensoring with $\mathbb{C}$).

   Write $V' = \mathbb{C} \otimes_{\mathbb{R}} V$ and let $0 \subsetneq U \subsetneq V'$ be a non-zero invariant complex subspace. As A' is semisimple, there is a (non-zero) invariant subspace X such that $W = U \oplus X$. But then $\dim_{\mathbb{C}} \mathrm{End}_{A'}(V') \geq 2$.

   On the other hand, every $\mathbb{C}$-linear map of V' is of the form $1 \otimes f + i \otimes g$, where $f, g \in \mathrm{End}_{A,\mathbb{R}}(V)$. Indeed, let $F \in \mathrm{End}_{A',\mathbb{C}}(V')$. Then $F(z \otimes v) = zF(1 \otimes v) = z(1 \otimes v' + i \otimes v'')$, and we set $f(v) = v'$, $g(v) = v''$. More abstractly, $\mathrm{End}_{A',\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{End}_{A,\mathbb{R}}(V)$. This shows $\dim_{\mathbb{C}} \mathrm{End}_{A'}(V') = \dim_{\mathbb{R}} \mathrm{End}_A(V) = 1$.

2. Since $\mathbb{C} \otimes 1$ in $\mathbb{C} \otimes_{\mathbb{R}} A$ commutes with $1 \otimes A$, it must map to $\mathrm{End}_{A,\mathbb{R}}(V) \cong \mathbb{C}$. Since unital algebra homomorphisms out of fields are injective (why?) this map is an isomorphism. Any two such maps differ by an ($\mathbb{R}$-linear) automorphism of $\mathbb{C}$. Write V and $\bar{V}$ for the two possibilities (the automorphism being id and complex conjugation, respectively). Any isomorphism of representations of $\mathbb{C} \otimes_{\mathbb{R}} A$ from $V \to \bar{V}$ are in particular real isomorphism of A-representations from V to V, hence given by multiplying with an element

from $\mathbb{C}$. As the automorphism given by complex conjugation is not inner (i.e. cannot be written as $z(-)z^{-1}$ for some $z \in \mathbb{C}$, the two representations V and $\bar{V}$ are not isomorphic as representations of $\mathbb{C} \otimes_{\mathbb{R}} A$.

3. The argument starts in the same way as above. Fix an automorphism $\mathrm{End}_{A,\mathbb{R}}(V) \cong \mathbb{H}$. If $\varphi : \mathbb{C} \to \mathbb{H}$ is an injective algebra map, we need to show that the representations V_φ for different choices of φ are equivalent as representations over $\mathbb{C}$.

   This follows if we can show that any two algebra maps $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{H}$ are related by conjugation with some $q \in \mathbb{H}^\times$, i.e. $\varphi_2(-) = q\varphi_1(-)q^{-1}$.

   To this end, note that $\varphi(i)^2 = -1$, and all elements $p \in \mathbb{H}$ with this property are of the form $ai+bj+ck$ with $a^2+b^2+c^2 = 1$ (to see this, write $p^2 = -1$ as $pp\bar{p} = -\bar{p}$ and use that $p\bar{p}$ is just the norm of p). Extending $\varphi(i)$ to an ON-basis in the span of $\{i, j, k\}$ shows that φ extends to an algebra isomorphism $\mathbb{H} \to \mathbb{H}$. By Exercise 43 all automorphism of $\mathbb{H}$ are inner, and so in particular φ_1, φ_2 differ by an inner automorphism.