Solutions for exercise sheet # 10
Topics in representation theory WS 2017

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Exercise 39

1. We can just copy the solution to exercise 38 part 2: Let $f : V \to A$ be as in the universal property of a Clifford algebra.

   Existence: By the universal property of the tensor algebra, we obtain an algebra homomorphism $\hat{f} : T(V) \to A$. Since $f(v)f(v) = -q(v)1$, $\hat{f}$ annihilates the ideal $\langle\, v \otimes v + q(v)1 \,|\, v \in V \,\rangle$ and hence factors through the quotient $Cl(V, q)$, giving an algebra homomorphism $\tilde{f} : Cl(V, q) \to A$ such that the required diagram commutes: $f = \tilde{f} \circ \iota$.

   Uniqueness: $Cl(V, q)$ is spanned as a K-vector spaces by products $\iota(v_1) \cdots \iota(v_k)$ of a finite number (including zero) of elements of V. But on each of these, $\tilde{f}$ is uniquely determined by being an algebra homomorphism and by the commuting diagram to be $\tilde{f}(\iota(v_1) \cdots \iota(v_k)) = f(v_1) \cdots f(v_k)$.

2. To see the composition property, consider the three commuting squares

$$\begin{array}{ccc} U & \xrightarrow{\iota} & Cl(U,p) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle Cl(f_2)} \\ V & \xrightarrow{\iota} & Cl(V,q) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle Cl(f_1)} \\ W & \xrightarrow{\iota} & Cl(W,r) \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{\iota} & Cl(U,p) \\ {\scriptstyle f_1\circ f_2}\downarrow & & \downarrow{\scriptstyle Cl(f_1\circ f_2)} \\ W & \xrightarrow{\iota} & Cl(W,r) \end{array}$$

   By the uniqueness in the universal property, we must have $Cl(f_1 \circ f_2) = Cl(f_1) \circ Cl(f_2)$.

Exercise 40

1. *Claim:* The centre of $\mathbb{H}$ is $\mathbb{R}1$.

   Proof: Let $z = a1 + bI + cJ + dK$ and suppose that $z \in Z(\mathbb{H})$. Since $Iz = aI1 + bII + cIJ + dIK = aI - b + cK - dJ$ and $zI = a1I + bII + cJI + dKI = aI - b - cK + dJ$, we must have $c = 0$ and $d = 0$. An analogous computation with J shows $b = 0$.

2. Let $x = \sum_{a=1}^{L} p_a \otimes q_a$ be minimal as in the hint. The p_a must be linearly independent, or we could write x as a shorter sum of pure tensors. We have $x' = x(1 \otimes q_1^{-1}) = \sum_{a=1}^{L} p_a \otimes \tilde{q}_a$, where $\tilde{q}_a = q_a q_1^{-1}$, so that $\tilde{q}_1 = 1$. As I is a two-sided ideal, $x' \in I$. Since multiplication by $1 \otimes q_1^{-1}$ is invertible, $x' \neq 0$.

Let now $y \in \mathbb{H}$ be arbitrary. The element $x'' = (1 \otimes y)x' - x'(1 \otimes y)$ is equally contained in I. Explicitly,

$$x'' = \sum_{a=1}^{L} p_a \otimes (y\tilde{q}_a - \tilde{q}_a y) = \sum_{a=2}^{L} p_a \otimes (y\tilde{q}_a - \tilde{q}_a y) \ ,$$

where in the second step we used that $\tilde{q}_1 = 1$. But by assumption, the shortest way to write a non-zero element in I in terms of pure tensors uses L terms. Hence x'' must be zero.

Since the p_a are linearly independent, we must have $y\tilde{q}_a - \tilde{q}_a y = 0$ for $a = 2, \dots, L$. As this holds for all y, the $\tilde{q}_a$ are central. By part 1 there exist $\lambda_a \in \mathbb{R}$ such that $q_a = \lambda_a 1$. Using this, we can rewrite x' as

$$x' = \sum_{a=1}^{L} p_a \otimes (\lambda_a 1) = \Big(\sum_{a=1}^{L} \lambda_a p_a \Big) \otimes 1 = p \otimes 1$$

for an appropriate $p \in \mathbb{H}$.

As $x' \neq 0$ we have $p \neq 0$. Multiplying by $p^{-1} \otimes 1$ gives $1 \otimes 1 \in I$. As this is the unit of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$, we have $I = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$.

Aside: Without modification, the same proof shows that for D a division algebra over a field k such that $Z(D) = k1$, the k-algebra $D \otimes_k D$ does not have non-trivial ideals. A more general version with a slightly different proof can be found e.g. in Theorem 3.5 and Lemma 3.7 in Farb, Dennis, Noncommutative Algebra (Springer, 1993).

Exercise 41

1. Write $1, I, J, K$ for the basis of $\mathbb{H}$.

   $Cl_{2,0}$: Consider the map $f : \mathbb{R}^2 \to \mathbb{H}$, $(x, y) \mapsto xI + yJ$. Then

   $$\begin{aligned} f(x,y)f(x,y) &= (xI + yJ)(xI + yJ) = -x^2 + xyIJ + xyJI - y^2 = -x^2 - y^2 \\ &= -q_{2,0}(x,y)1 \ . \end{aligned}$$

   By the universal property we obtain an algebra homomorphism $\tilde{f} : Cl_{2,0} \to \mathbb{H}$. $\tilde{f}$ maps $1 \mapsto 1$, $e_1 \mapsto I$, $e_2 \mapsto J$, $e_1 e_2 \mapsto IJ = K$. Hence it maps an $\mathbb{R}$-basis to an $\mathbb{R}$-basis and therefore is an isomorphism.

   $Cl_{2,0}$: Consider the map $f : \mathbb{R}^2 \to \mathrm{Mat}(2, \mathbb{R})$, $(x, y) \mapsto \left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right)$. Then

   $$\begin{aligned} f(x,y)f(x,y) &= \left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{smallmatrix}\right) = (x^2 + y^2)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= -q_{0,2}(x,y)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \ . \end{aligned}$$

   The rest of the argument is as above.

2. The map $x \mapsto px\bar{q}$ is $\mathbb{R}$-linear, as required. The map ψ is $\mathbb{R}$-linear, too. $\psi(1 \otimes 1) = (x \mapsto x) = id_{\mathbb{H}}$, so ψ is unital and

$$\psi((p \otimes q)(r \otimes s))(x) = \psi((pr) \otimes (qs))(x) = (pr)x\overline{(qs)} = p\,r\,x\,\bar{s}\,\bar{q}\ ,$$

as well as

$$(\psi(p \otimes q) \circ \psi(r \otimes s))(x) = \psi(p \otimes q)(rx\bar{s}) = p\,r\,x\,\bar{s}\,\bar{q}\ ,$$

so that ψ is compatible with the product. Altogether, this shows that ψ is an $\mathbb{R}$-algebra homomorphism.

It remains to show bijectivity.

Elementary method: Pairing elements of $\{1, I, J, K\}$ with each other gives a 16-element basis of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. ψ applied to each of these 16 elements can be evaluated on that same basis, giving a 4×4 matrix. This produces 16 4×4 matrix (with entries $0, \pm 1$) which can then be checked to be linearly independent.

Abstract method: We have seen in exercise 40 that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ has no non-trivial two-sided ideals. Thus the kernel of ψ is either 0 or $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. As $\psi(1 \otimes 1) = id_{\mathbb{H}} \neq 0$, the kernel is not $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. Thus ψ is injective. Source and target of ψ have the sime dimension as $\mathbb{R}$-vector spaces. Hence ψ is also surjective.

Exercise 42

(All references are to section 3.3.) The starting point is lemma 1 (and the trivial case of a 0-dim v.sp.):

$$Cl_{0,0} \cong \mathbb{R}\ \ ,\ \ Cl_{1,0} \cong \mathbb{C}\ \ ,\ \ Cl_{2,0} \cong \mathbb{H}\ \ ,\ \ Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}\ \ ,\ \ Cl_{0,2} \cong \mathrm{Mat}(2, \mathbb{R})\ .$$

From lemma 3 we learn

$$Cl_{n,0} \cong Cl_{0,n-2} \otimes Cl_{2,0} \cong Cl_{0,n-2} \otimes \mathbb{H}\ ,$$
$$Cl_{0,n} \cong Cl_{n-2,0} \otimes Cl_{0,2} \cong Cl_{n-2,0} \otimes \mathrm{Mat}(2, \mathbb{R})\ .$$

From this we can determine more entries in the zero's row and column

$$Cl_{3,0} \cong Cl_{0,1} \otimes \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H} \qquad Cl_{4,0} \cong \mathrm{Mat}(2, \mathbb{H})$$
$$Cl_{0,3} \cong Cl_{1,0} \otimes \mathrm{Mat}(2, \mathbb{R}) \cong \mathrm{Mat}(2, \mathbb{C}) \qquad Cl_{0,4} \cong \mathrm{Mat}(2, \mathbb{H})$$

For the remaining entries in the zero's row and column we iterate the above relation:

$$Cl_{n,0} \cong Cl_{0,n-2} \otimes \mathbb{H} \cong Cl_{n-4,0} \otimes \mathbb{H} \otimes \mathrm{Mat}(2, \mathbb{R})\ ,$$
$$Cl_{0,n} \cong Cl_{0,n-4} \otimes \mathbb{H} \otimes \mathrm{Mat}(2, \mathbb{R})\ .$$

We can now determine the remaining of the first 8 entries in the 0-row and column, using also lemma 5:

$Cl_{5,0} \cong Cl_{1,0} \otimes \mathbb{H} \otimes \mathrm{Mat}(2,\mathbb{R}) \cong \mathbb{C} \otimes \mathbb{H} \otimes \mathrm{Mat}(2,\mathbb{R}) \cong \mathrm{Mat}(2,\mathbb{C}) \otimes \mathrm{Mat}(2,\mathbb{R}) \cong \mathrm{Mat}(4,\mathbb{C})$
$Cl_{6,0} \cong \mathrm{Mat}(8,\mathbb{R})$
$Cl_{7,0} \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{H} \otimes \mathrm{Mat}(2,\mathbb{R}) \cong \mathrm{Mat}(4,\mathbb{R})^{\oplus 2} \otimes \mathrm{Mat}(2,\mathbb{R}) \cong \mathrm{Mat}(8,\mathbb{R})^{\oplus 2}$
$Cl_{0,5} \cong \mathrm{Mat}(2,\mathbb{H})^{\oplus 2}$
$Cl_{0,6} \cong \mathrm{Mat}(4,\mathbb{H})$
$Cl_{0,7} \cong \mathrm{Mat}(8,\mathbb{C})$

These entries agree with the table in theorem 6. Iterating lemma 3 once more gives

$$Cl_{n,0} \cong Cl_{n-8,0} \otimes \mathrm{Mat}(16,\mathbb{R}) \quad , \quad Cl_{0,n} \cong Cl_{0,n-8} \otimes \mathrm{Mat}(16,\mathbb{R}) \ .$$

Or, equivalently, for $k \geq 0$,

$$Cl_{n+8k,0} \cong Cl_{n,0} \otimes \mathrm{Mat}(2^{8k/2},\mathbb{R}) \quad , \quad Cl_{0,n+8k} \cong Cl_{0,n} \otimes \mathrm{Mat}(2^{8k/2},\mathbb{R}) \ .$$

This establishes the table for all $Cl_{d,0}$ and $Cl_{0,d}$.
Finally, once more by lemmas 1 and 3, $Cl_{r+1,s+1} \cong Cl_{r,s} \otimes \mathrm{Mat}(2,\mathbb{R})$, that is, for $k \geq 0$,

$$Cl_{r+k,s+k} \cong Cl_{r,s} \otimes \mathrm{Mat}(2^k,\mathbb{R}) \ .$$

This completes the proof of theorem 6.

Exercise 42

1. An $\mathbb{H}$-$\mathbb{H}$-bimodule is the same as an $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{op}$ left module. The conjugation $\overline{(\ )}$ gives an $\mathbb{R}$-algebra isomorphism $\mathbb{H} \to \mathbb{H}^{op}$. Hence $\mathbb{H}$ is an $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$-left module. By exercise 40, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}(4,\mathbb{R})$. By theorem 1.2.9, $\mathrm{Mat}(4,\mathbb{R})$ is semisimple and has the unique simple module $\mathbb{R}^4$.

2. Let $f : \mathbb{H} \to {}_{\alpha}\mathbb{H}$ be the bimodule isomorphism. Compatibility with the left action means, for all $p, q \in \mathbb{H}$, $f(pq) = \alpha(p)f(q)$. Compatibility with the right action means $f(pq) = f(p)q$. Evaluating the first for $q = 1$ and the second for $p = 1$ shows $f(p) = \alpha(p)f(1)$ and $f(q) = f(1)q$. Thus

$$\alpha(p)f(1) = f(p) = f(1)p \ .$$

Now $f(1) \neq 0$ (or f would not be invertible) and so $\alpha(p) = f(1)pf(1)^{-1}$.