Solutions for exercise sheet # 09
Topics in representation theory WS 2017

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Exercise 35

Let A be a k-algebra and let $f : V \to A$ be linear.

Existence: Define the map $f_n : V^{\times n} \to A$, $f_n(v_1, \dots, v_n) = f(v_1) \cdots f(v_n) \in A$. This map is multilinear and hence there is a unique linear $\tilde{f}_n : V^{\otimes n} \to A$ such that $\tilde{f}_n(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$. Define $f_0 : k \to A$ via $\tilde{f}_0(1) = 1$. Define $\tilde{f} : T(V) \to A$ as $f|_{T^n} = f_n$. By construction, $f = \tilde{f} \circ i$. Furthermore, $\tilde{f}$ is an algebra homomorphism. Indeed, it is enough to verify this on pure tensors (as they span $T(V)$, and we have, for $x = x_1 \otimes \cdots \otimes x_a$ and $y = y_1 \otimes \cdots \otimes y_b$,

$$\tilde{f}(xy) = \tilde{f}(x_1 \otimes \cdots \otimes x_a \otimes y_1 \otimes \cdots \otimes y_b) = f(x_1) \cdots f(x_a) f(y_1) \cdots f(y_b) = \tilde{f}(x)\tilde{f}(y) \ .$$

Finally, again by construction $\tilde{f}(1) = 1$.

Uniqueness: $\tilde{f}$ is determined by its value on pure tensors. In order to be an algebra homomorphism, we must have $\tilde{f}(1) = 1$ and $\tilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \cdots f(v_n)$. Hence $\tilde{f}$ is uniquely determine by f.

Exercise 36

1. Consider the map $f : V \to T(V) \otimes T(V)$ given by $f(v) = i(v) \otimes 1 + 1 \otimes i(v)$. By the universal property, there is a unique algebra map $\Delta : T(V) \to T(V) \otimes T(V)$ with the stated property.

2. We have (omitting all "i"):

$$\begin{aligned}\Delta(xy) = \Delta(x)\Delta(y) &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) \\ &= (x \otimes y) \otimes 1 + x \otimes y + y \otimes x + 1 \otimes (x \otimes y) \ ,\end{aligned}$$

   where in the last line, the summands lie in $T^2 \otimes T^0$, $T^1 \otimes T^1$, $T^1 \otimes T^1$, $T^0 \otimes T^2$, respectively.

3. This follows from uniqueness in the universal property. Both, $L := (id \otimes \Delta) \circ \Delta$ and $R := (\Delta \otimes id) \circ \Delta$ are algebra homomorphism $T(V) \to T(V) \otimes T(V) \otimes T(V)$. They furthermore satisfy

$$L(i(v)) = (id \otimes \Delta)(v \otimes 1 + 1 \otimes v) = v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v$$

   and

$$R(i(v)) = (\Delta \otimes id)(v \otimes 1 + 1 \otimes v) = v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v \ .$$

   Thus $L \circ i = R \circ i$, and so by the universal property of the tensor algebra, $L = R$.

Exercise 37

1. *Existence:* We need to define a product on A/I. Our ansatz is $(a+I)(b+I) := ab+I$. To see that this is independent of the choice of representatives a, b, let a', b' be such that $a'+I = a+I$ and $b'+I = b+I$. Equivalently, $a'-a \in I$ and $b'-b \in I$. Then

$$a'b'+I = (a+a'-a)(b+b'-b)+I = ab+(a'-a)b+a(b'-b)+(a'-a)(b'-b)+I = ab+I\ ,$$

where in the last step we used that $a'-a, b'-b \in I$ and that I is a two-sided ideal.

Associativity of the product on A implies that of the just defined product on A/I. The unit is $1+I$.

Note that by definition, $\pi(ab) = \pi(a)\pi(b)$ and $\pi(1) = 1+I$.

Uniqueness: Let $\mu : A/I \otimes A/I \to A/I$ be an associative unital product on A with unit e, such that π is an algebra homomorphism.

Since π is unital, we have $e = \pi(1) = 1+I$, so the unit is unique. For the product, let $\alpha, \beta \in A/I$ be arbitrary. Since π is surjective, there are $a, b \in A$ such that $\alpha = \pi(a)$, $\beta = \pi(b)$. Then $\mu(\alpha \otimes \beta) = \mu(\pi(a) \otimes \pi(b)) = \pi(ab) = (a+I)(b+I)$, where in the last step we used the product constructed in the existence part of the argument. Thus the product is uniquely determined.

2. With the notation in the diagram:

Claim (universal property): For each K-algebra B and algebra homomorphism $f : A \to B$ such that $f(I) = 0$, there exists a unique algebra homomorphism $\tilde{f} : A/I \to B$ such that $\tilde{f} \circ \pi = f$.

Proof: We first show that there is a unique linear map making the diagram commute, and then that this linear map is necessarily an algebra homomorphism.

Existence and uniqueness of the linear map follows from the construction of linear maps out of quotient spaces: as f vanishes on the K-linear subspace I, it descends to a K-linear map $A/I \to B$.

Now $\tilde{f}(1+I) = \tilde{f} \circ \pi(1) = f(1) = 1$, so that $\tilde{f}$ preserves the unit. Furthermore, $\tilde{f}((a+I)(b+I)) = \tilde{f}(\pi(a)\pi(b)) = \tilde{f}(\pi(ab)) = f(ab) = f(a)f(b) = \tilde{f}(\pi(a))\tilde{f}(\pi(b)) = \tilde{f}(a+I)\tilde{f}(b+I)$, so that $\tilde{f}$ is compatible with the products in A/I and B.

Exercise 38

1. Let V be a K-vector space. Let (Λ, λ) and (Λ', λ') be two alternating algebras for V. The universal property of Λ, applied to the linear map $\lambda' : V \to \Lambda'(V)$

(which satisfies $\lambda'(v)\lambda'(v) = 0$ by definition) yields an algebra homomorphism $\tilde{\lambda}' : \Lambda \to \Lambda'$, such that $\tilde{\lambda}' \circ \lambda = \lambda'$. Conversely, one obtains an algebra homomorphism $\tilde{\lambda} : \Lambda' \to \Lambda$ such that $\tilde{\lambda} \circ \lambda' = \lambda$.

Thus also $\tilde{\lambda} \circ \tilde{\lambda}' : \Lambda \to \Lambda$ is an algebra homomorphism. It satisfies $\tilde{\lambda} \circ \tilde{\lambda}' \circ \lambda = \tilde{\lambda} \circ \lambda' = \lambda$. But also the identity on Λ satisfies $id_\Lambda \circ \lambda = \lambda$. By uniqueness in the universal property, we must have $\tilde{\lambda} \circ \tilde{\lambda}' = id_\Lambda$. Similarly one sees that $\tilde{\lambda}' \circ \tilde{\lambda} = id_{\Lambda'}$.

By construction, the $\tilde{\lambda}$ and $\tilde{\lambda}'$ are the unique isomorphisms compatible with the maps λ, λ' in that $\tilde{\lambda} \circ \lambda' = \lambda$ and $\tilde{\lambda}' \circ \lambda = \lambda'$ (draw the diagrams for more clarity).

2. Denote the ideal defining $\Lambda(V)$ by $J = \langle v \otimes v \,|\, v \in V \rangle$. Let $f : V \to A$ be as in the universal property of an alternating algebra.

   Existence: By the universal property of the tensor algebra, we obtain an algebra homomorphism $\hat{f} : T(V) \to A$. Since $f(v)f(v) = 0$, $\hat{f}$ annihilates the ideal J and hence factors through the quotient $\Lambda(V)$, giving an algebra homomorphism $\tilde{f} : \Lambda(V) \to A$ such that the required diagram commutes: $f = \tilde{f} \circ i_\Lambda$.

   Uniqueness: $\Lambda(V)$ is spanned as a K-vector spaces by products $i_\Lambda(v_1) \cdots i_\Lambda(v_k)$ of a finite number (including zero) of elements of V. But on each of these, $\tilde{f}$ is uniquely determined by being an algebra homomorphism and by the commuting diagram to be $\tilde{f}(i_\Lambda(v_1) \cdots i_\Lambda(v_k)) = f(v_1) \cdots f(v_k)$.

3. It is clear that $\Lambda(V)$ is spanned by the Λ^m. It remains to show that the sum is direct. Write $\Lambda^{\neq m} = \mathrm{span}(\Lambda^n | n \neq m) = \pi(\bigoplus_{n \neq m} T^n)$. We need to show that $\Lambda^m \cap \Lambda^{\neq m} = \{0\}$.

   Let $u \in \Lambda^m \cap \Lambda^{\neq m}$. Then there are $x \in T^m$ and $y \in \bigoplus_{n \neq m} T^n$ such that $u = x + J$ and $u = y + J$. Thus $x - y \in J$. We can therefore find homogeneous elements $a_r, b_r \in T^r$, $v_r \in V$ such that

   $$x - y = \sum_r a_r \otimes v_r \otimes v_r \otimes b_r \ .$$

   We can now split this into two sums by degree: Write $|a_r|$ for the degree of a_r. Then

   $$x - y = \sum_{r, |a_r| + |b_r| + 2 = m} a_r \otimes v_r \otimes v_r \otimes b_r + \sum_{r, |a_r| + |b_r| + 2 \neq m} a_r \otimes v_r \otimes v_r \otimes b_r \ .$$

   But $T(V)$ is a direct sum of the T^k, $k = 0, 1, \ldots$, and for the above equality to hold, we must have

   $$x = \sum_{r, |a_r| + |b_r| + 2 = m} a_r \otimes v_r \otimes v_r \otimes b_r \quad , \quad y = - \sum_{r, |a_r| + |b_r| + 2 \neq m} a_r \otimes v_r \otimes v_r \otimes b_r \ .$$

   In particular, $x \in J$ and $y \in J$, and hence $u = x + J = J$, which means $u = 0$ in $\Lambda(V)$.

4. We will define maps $F : S_\lambda(V) \to \Lambda^m$ and $G : \Lambda^m \to S_\lambda(V)$ and show that they are inverse to each other.

The map F: The vector space $S_\lambda(V)$ is the subspace of $V^{\otimes m}$ spanned by elements

$$\sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \quad , \quad v_1, v_2, \ldots, v_m \in V \ .$$

We define F to be the restriction of the canonical projection $\pi : V^{\otimes m} \to \Lambda^m$ to $S_\lambda V$.

The map G: We first define a linear map $g : V^{\otimes m} \to S_\lambda V$ and then show that it descends to a map $G : \Lambda^m \to S_\lambda V$. The map g is simply the normalised symmetriser:

$$g(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \ .$$

Let $x = \sum_r a_r \otimes v_r \otimes v_r \otimes b_r$ be an arbitrary homogeneous element of J of degree m. We need to show that $g(x) = 0$. We may assume that a_r and b_r are pure tensors, too. In fact, already $g(a_r \otimes v_r \otimes v_r \otimes b_r) = 0$ before taking the sum. To see this, let τ be the transposition which exchanges the two factors containing v_r. Then $(a_r \otimes v_r \otimes v_r \otimes b_r).\sigma = (a_r \otimes v_r \otimes v_r \otimes b_r).\sigma\tau$, but these two terms contribute to the sum over S_m with opposite signs.

Hence there is a linear map $G : \Lambda^m \to S_\lambda V$ such that $G(v_1 \otimes \cdots \otimes v_m + J_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$. Here, J_m is the degree m subspace of J.

$F \circ G = id$: We have

$$F(G(v_1 \otimes \cdots \otimes v_m + J_m)) = F(g(v_1 \otimes \cdots \otimes v_m)) = v_1 \otimes \cdots \otimes v_m + J_m \ ,$$

since every of the $m!$ summands in $g(v_1 \otimes \cdots \otimes v_m)$ can be reordered in Λ^m to $v_1 \otimes \cdots \otimes v_m + J_m$, cancelling the sign in the sum.

$G \circ F = id$: Note that elements of the form $g(v_1 \otimes \cdots \otimes v_m)$ span $S_\lambda V$, and that $g \circ g = g$. By the same reordering argument as above, we compute

$$G(F(g(v_1 \otimes \cdots \otimes v_m))) = G(v_1 \otimes \cdots \otimes v_m + J_m) = g(v_1 \otimes \cdots \otimes v_m) \ .$$

$GL(V)$-action: Since the $GL(V)$-action leaves the summands J_m of the ideal J invariant, we get a well-defined action on Λ^m, that is, $\pi : V^{\otimes m} \to \Lambda^m$ becomes a $GL(V)$-intertwiner. Thus, so is the restriction F of π to $S_\lambda V$, and $S_\lambda V \cong \Lambda^m$ as $GL(V)$-modules.