Solutions for exercise sheet # 08
Topics in representation theory WS 2017

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Exercise 30

1. This is simply the transformation formula for integrals: $\tilde{\Lambda}$ is a diffeomorphism from $\mathbb{R}^3$ to itself, and so for all integrable functions $g : \mathbb{R}^3 \to \mathbb{C}$ we have
$$\int_{\mathbb{R}^3} g(x) d^3x = \int_{\mathbb{R}^3} g(\tilde{\Lambda} y)\, |\det D\tilde{\Lambda}(y)|\, d^3y \ .$$
Thus
$$\begin{aligned} I(f \circ \tilde{\Lambda}) &= \int_{\mathbb{R}^3} f(\tilde{\Lambda}(y))\mu(y) d^3y \overset{(*)}{=} \int_{\mathbb{R}^3} f(\tilde{\Lambda}(y))\mu(\tilde{\Lambda}(y))\, |\det D\tilde{\Lambda}(y)|\, d^3y \\ &\overset{(**)}{=} \int_{\mathbb{R}^3} f(x)\mu(x) d^3x = I(f) \ . \end{aligned}$$
Here, (*) is the assumed property of μ in the problem and (**) the above transformation of the integral, applied to $g(x) = f(x)\mu(x)$.

2. For a rotation $R \in SO(3)$ let $\Lambda \in SO(1,3)$ be the corresponding element in the Lorenz group (a block matrix with 1 in the upper left corner). Then $\tilde{\Lambda} = R$ and so $\det \tilde{\Lambda} = \det R = 1$. Since $\mu(\vec{p})$ only depends on $|\vec{p}|$ also $\mu(\tilde{\Lambda}\vec{p}) = \mu(\vec{p})$.
Consider now a boost. Let $s = \sinh(\theta)$, $c = \cosh(\theta)$ and
$$\Lambda = \begin{pmatrix} c & s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
Let $\vec{p} \in \mathbb{R}^3$ and write $p = \pi^{-1}(\vec{p}) = (p_0, \vec{p})$ with $p_0 = \sqrt{m^2 + |\vec{p}|^2}$. Then
$$\Lambda(p) = (cp_0 + sp_1, sp_0 + cp_1, p_2, p_3)$$
and
$$\tilde{\Lambda}(\vec{p}) = (sp_0 + cp_1, p_2, p_3) = (s\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2} + cp_1, p_2, p_3) \ .$$
Now $(D\tilde{\Lambda}(\vec{p}))_{ij} = \frac{\partial}{\partial p_j}\tilde{\Lambda}_i(\vec{p})$. Note that $\frac{\partial}{\partial p_j} p_0 = p_j/p_0$. Thus
$$D\tilde{\Lambda}(\vec{p}) = \begin{pmatrix} sp_1/p_0 + c & sp_2 & sp_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \det \tilde{\Lambda}(\vec{p}) = s\frac{p_1}{p_0} + c \ .$$
Finally,
$$\frac{\mu(\vec{p})}{\mu(\tilde{\Lambda}(\vec{p})} = \frac{(\Lambda p)_0}{p_0} = \frac{cp_0 + sp_1}{p_0} \ ,$$

which establishes the transformation rule for boosts.

By exercise 17, every element of $L_+^\uparrow$ is a product (rotation)(boost as a above)(rotation). Since $\Lambda \mapsto \tilde{\Lambda}$ is a group homomorphism and the determinant is multiplicative, this proves the lemma.

3. Since $SO(1,3)$ acts transitively on $\mathbb{R}^3$, (*) determines $\mu(\vec{p})$ uniquely in terms of $\mu(0)$. Hence all possible choices of μ agree up to a scalar factor (namely the value at 0).

Exercise 31

We have

$$mid + \hat{q} = \begin{pmatrix} m + q_0 + q_3 & q_1 - iq_2 \\ q_1 + iq_2 & m + q_0 - q_3 \end{pmatrix}$$

so that

$$\det(mid + \hat{q}) = (m+q_0)^2 - q_3^2 - q_1^2 - q_2^2 = m^2 + 2mq_0 + m^2 = 2m(m+q_0) \ .$$

This shows $\det T_q = 1$.

Since $\hat{a} = mid$, we need to compute

$$T_q \hat{a} T_q^\dagger = T_q T_q = m\frac{(mid + \hat{q})^2}{2m(m+q_0)} = \frac{m^2 id + 2m\hat{q} + \hat{q}^2}{2(m+q_0)}$$

A short calculation using $q_0^2 - q_1^2 - q_2^2 - q_3^2 = m^2$ gives

$$\hat{q}^2 = -m^2 id + 2q_0 \hat{q} \ ,$$

which shows that $T_q \hat{a} T_q^\dagger = \hat{q}$.

Exercise 32

We will not write π and π^{-1}. If an element of $\mathbb{R}^3$ serves as an argument for a function expecting $\mathbb{R}^4$, the application of π^{-1} is understood, and vice versa.

We abbreviate, for $p \in \mathbb{R}^3$, $g \in SL(2,\mathbb{C})$, $R(p,g) = W(\phi(g)^{-1}p, g) = T_p^{-1} g T_{\phi(g^{-1})p}$.

Let R be a finite dimensional unitary representation of the little group $SU(2)$.

On $L^2(\mathbb{R}^3, R)$ the action is

$$\big((x,g)\psi\big)(p) = e^{i\eta(x,p)} R(p,g) \psi(\widetilde{\phi(g^{-1})}p)$$

Then

$$\begin{aligned}
\big((x,g)((y,h)\psi)\big)(p) &= e^{i\eta(x,p)} R(p,g)((y,h)\psi)(\widetilde{\phi(g^{-1})}p) \\
&= e^{i\eta(x,p)} R(p,g) e^{i\eta(y,\widetilde{\phi(g^{-1})}p)} R(\widetilde{\phi(g^{-1})}p, h) \psi(\widetilde{\phi(h^{-1})}\widetilde{\phi(g^{-1})}p) \\
&= e^{i\eta(x,p)} e^{i\eta(\phi(g)y,p)} R(p,g) R(\widetilde{\phi(g^{-1})}p, h) \psi(\phi(\widetilde{(gh)}^{-1})p)
\end{aligned}$$

Note that

$$R(p,g)R(\widetilde{\phi(g^{-1})}p,h) = T_p^{-1} g T_{\phi(g^{-1})p} T^{-1}_{\phi(g^{-1})p} h T_{\phi(h^{-1})\phi(g^{-1})p} = R(p,gh) \ .$$

Thus altogether

$$\big((x,g)((y,h)\psi)\big)(p) = e^{i\eta(x+\phi(g)y,p)} R(p,gh)\psi(\phi(\widetilde{(gh)}^{-1})p) \ .$$

Since $(x,g)(y,h) = (x+\phi(g)y, gh)$, and since the right hand side is equal to $((x+\phi(g)y, gh)\psi)(p)$, this shows the claim.

Exercise 33

1. For the given choice of a we have $\widehat{a} = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. The left hand side of the condition $M\hat{a}M^\dagger = \hat{a}$ reads

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar\alpha & \bar\gamma\\ \bar\beta & \bar\delta\end{pmatrix} = \begin{pmatrix}\alpha\bar\alpha & \alpha\bar\gamma\\ \gamma\bar\alpha & \gamma\bar\gamma\end{pmatrix} \ .$$

   Thus $M\hat{a}M^\dagger = \hat{a}$ is equivalent to $|\alpha| = 1$ and $\gamma = 0$. Since $M \in SL(2,\mathbb{C})$ we also have $\alpha\delta = 1$, i.e. $\delta = \bar\alpha$. This shows the claim.

2. That τ is a surjective is clear. For the group homomorphism property note that

$$\begin{aligned}&\begin{pmatrix}e^{i\theta/2} & ze^{-i\theta/2}\\ 0 & e^{-i\theta/2}\end{pmatrix}\begin{pmatrix}e^{i\theta'/2} & z'e^{-i\theta'/2}\\ 0 & e^{-i\theta'/2}\end{pmatrix}\\ &= \begin{pmatrix}e^{i(\theta+\theta')/2} & z'e^{i(\theta-\theta')/2} + ze^{-i(\theta+\theta')/2}\\ 0 & e^{-i(\theta+\theta')/2}\end{pmatrix}\\ &= \begin{pmatrix}e^{i(\theta+\theta')/2} & e^{-i(\theta+\theta')/2}(z+z'e^{i\theta})\\ 0 & e^{-i(\theta+\theta')/2}\end{pmatrix} \ .\end{aligned}$$

   The kernel is $\{\pm id\}$.

Exercise 34

1. For $\mathcal{O}_m^+$ and $\mathcal{O}_0^+$ (orbits 1 and 4 in our list), this was already done in the lecture. In both cases, the little group did allow for finite-dimensional unitary irreps.

2. *Orbit 2*: Let $m > 0$ and choose $a = (-m,0,0,0)$. Since the stabiliser is of $m\,id_{2\times2}$ is the same as that of $-m\,id_{2\times2}$, the result is the same as for $\mathcal{O}_m^+$: The little group is $SU(2)$, and does allow for finite-dimensional representations.

3. *Orbit 6*: We may take $a = (-\frac{1}{2},0,0,-\frac{1}{2})$, for which $\widehat{a} = \left(\begin{smallmatrix}-1&0\\0&0\end{smallmatrix}\right)$. Again the stabiliser of a and $-a$ are the same, and the resulting little group is the same as for $\mathcal{O}_0^+$.

4. *Orbit 3*: Pick $a = (0, 0, m, 0)$ for $m > 0$. Then $\widehat{a} = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Thus $M \in \mathrm{Stab}(a)$ if and only if (we can cancel the factor of i)

$$M \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Writing this out in components for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ gives

$$\begin{pmatrix} 2i\mathrm{Im}(\bar{a}b) & b\bar{c} - a\bar{d} \\ \bar{a}d - \bar{b}c & 2i\mathrm{Im}(\bar{c}d) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Writing a, b, c, d in "signed polar coordinates", i.e. $z = xe^{i\theta}$ with $\theta \in [0, \pi)$ and $x \in \mathbb{R}$, we see that the diagonal entries force a, b to have the same phase, say $a = re^{i\theta}$, $b = se^{i\theta}$, and similarly $c = te^{i\psi}$, $b = ue^{i\psi}$. The off-diagonal conditions then further force $\theta = \psi$, and the fact that $M \in SL(2, \mathbb{C})$, i.e. that $ad - bc = 1$, forces $\theta = 0$. Thus a, b, c, d are in fact real, and so must be in $SL(2, \mathbb{R})$. Conversely, every element of $SL(2, \mathbb{R})$ solves the above condition.

The same argument used for $SL(2, \mathbb{C})$ shows that $SL(2, \mathbb{R})$ does not have non-trivial finite-dimensional unitary representations.

5. *Orbit 5*: The stabiliser is all of $\tilde{P}$, whose only finite-dimensional unitary representations are the ones with trivial action.