Solutions for exercise sheet # 07
Topics in representation theory WS 2017

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Exercise 26

1. L is already a subgroup of $GL(4,\mathbb{R})$ and we take this as the action of L on $\mathbb{R}^4$. The product of $\mathbb{R}^r \rtimes L$ is then
$$(v,\Lambda)(w,\Gamma) = (v + \Lambda(w), \Lambda\Gamma) \ .$$
Consider the map
$$f : \mathbb{R}^r \rtimes L \longrightarrow P \quad , \quad (v,\Lambda) \longmapsto (x \mapsto \Lambda x + v) \ .$$
This map is a bijection by the definition of P. We check that it is a group homomorphism. We have
$$f((v,\Lambda)(w,\Gamma)) = f(v + \Lambda(w), \Lambda\Gamma) = \big(x \mapsto \Lambda\Gamma(x) + v + \Lambda(w)\big)$$
and
$$\begin{aligned} f(v,\Lambda) \circ f(w,\Gamma) &= \big(x \mapsto (\Lambda(-) + v)(\Gamma(x) + w)\big) = \big(x \mapsto (\Lambda(\Gamma(x) + w) + v)\big) \\ &= \big(x \mapsto \Lambda\Gamma(x) + \Lambda(w) + v\big) \ , \end{aligned}$$
which agrees.

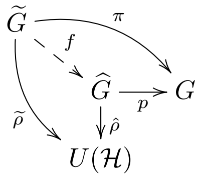
2. (a) Write $h : \mathbb{R}^4 \to H$ for the map $\widehat{(\ )}$ from the lecture. Then for $v \in \mathbb{R}^4$, $\phi(M)(v) = h^{-1}(Mh(v)M^\dagger)$. Using $\det(h(v)) = \eta(v,v)$ we compute, for all $v \in \mathbb{R}^4$,
$$\eta(\phi(M)v, \phi(M)v) = \det(h(\phi(M)v)) = \det(Mh(v)M^\dagger) = \det(h(v)) = \eta(v,v) \ .$$
Applying this to $v = x + y$ shows that also $\eta(\phi(M)x, \phi(M)y) = \eta(x,y)$. Thus $\phi(M) \in L$.
From Exercise 11 we know that $SL(2,\mathbb{C})$ is connected. The map h is continuous, and so the image of $SL(2,\mathbb{C})$ under h is connected, too. Since $\phi(id) = id$, the image of h lies in the connected component of id in L, which is $L_+^\uparrow$.

   (b) We compute
$$(\phi(M)\phi(N))(v) = h^{-1}(Mh(\phi(N)(v))M^\dagger) = h^{-1}(MNh(v)N^\dagger M^\dagger) = \phi(MN)(v)$$

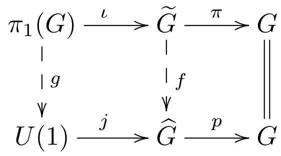
   (c) Suppose $M \in SL(2,\mathbb{C})$ satisfies $\phi(M)(v) = v$ for all $v \in \mathbb{R}^3$. This is equivalent to $MXM^\dagger = X$ for all hermitian matrices X. Setting $X = id$ shows that $MM^\dagger = id$, i.e. that $M^\dagger = M^{-1}$. But then $MXM^\dagger = X$ is equivalent to $MX = XM$. This condition is $\mathbb{C}$-linear in X, and so it holds for all hermitian X iff it holds for all complex 2×2 matrices X. The centre of $\mathrm{Mat}(2,\mathbb{C})$ is λid. Since $M \in SL(2,\mathbb{C})$ and $M^{-1} = M^\dagger$, we see that $M \in SU(2)$. The only diagonal matrices in $SU(2)$ are $\pm id$.

Exercise 27

The universal property of pullback squares (Lemma 2.4.1) we obtain a group homomorphism f such that



commutes. We obtain a group homomorphism $g : \pi_1(G) \to U(1)$ such that



commutes since $p \circ f \circ \iota = 1$, and so $f \circ \iota$ factors through $U(1)$.

Claim: $g = \zeta$

Since j is injective, g is the unique map such that $jg = f\iota$. Denoting the embedding $U(1) \to U(\mathcal{H})$ by ε, the map ζ is the unique one such that $\varepsilon\zeta = \widetilde{\rho}\iota$. But $\widetilde{\rho} = \widehat{\rho} f$ by construction of f and so $\varepsilon\zeta = \widehat{\rho} f \iota = \widehat{\rho} j g$. Now $\widehat{\rho} j = \varepsilon$ by the first commuting diagram in the statement of the exercise, so finally $\varepsilon\zeta = \varepsilon j$ which by injectivity of ε implies $\zeta = g$.

Claim: f is not surjective.
Indeed, suppose f is surjective. Let $u \in \widehat{G}$ lie in the image of $U(1)$. Let $x \in \widetilde{G}$ be such that $u = f(x)$. Then $p(f(x)) = 1$, hence $\pi(x) = 1$. Therefore $x \in \pi_1(G)$, and so $u = g(x)$. But $\pi_1(G)$ is at most countable and $U(1)$ is not, so there cannot be a surjection $\pi_1(G) \to U(1)$.

Claim: f is injective iff g is injective.
Suppose f is injective. Let $x \in \pi_1(G)$ satisfy $g(x) = 1$. Then also $j(g(x)) = 1$ and hence $f(\iota(x)) = 1$. But f and ι are injective, and so $x = 1$. Hence g is injective.
Conversely, suppose g is injective. Let $y \in \widetilde{G}$ satisfy $f(y) = 1$. Then also $f(p(y)) = 1$, hence $\pi(y) = 1$ and so $y = \iota(x)$ for some $x \in \pi_1(G)$. But then $1 = f(y) = f(\iota(x)) = j(g(x))$ and j and g are injective. So $x = 1$ and thereby also $y = 1$.

Altogether we see that f is injective iff ρ is such that every element of $\pi_1(G)$ acts non-trivially on $U(\mathcal{H})$.

Exercise 28

1. The implications follow from the bounds $\|(f(x)-f(y))v\| \leq \|f(x)-f(y)\|\,\|v\|$ (by the definition of the operator norm as a supremum) and $|(u,(f(x)-f(y))v)| \leq \|u\|\|(f(x)-f(y))v\|$ (Cauchy ineuality).

2. Claim: f is weakly continuous

   Let $u = \sum_n u_n e_n$ and dito for v. Since $\sum_n |u_n|^2$ converges, the u_n go to zero for $n \to \infty$. Thus, for $n^{-1} \to 0$,

   $$(u, f(\tfrac{1}{n})v) = (u, \sum_k v_k e_{k+n}) = \sum_k \bar{u}_{k+n} v_k = (L_n u, v) \leq \|L_n u\|\,\|v\| \ ,$$

   where L_n is the left shift. But $\|u\|^2$ is the norm-squared of the first $n-1$ components of u with $\|L_n u\|^2$, which shows that $\|L_n u\| \to 0$ for $n \to \infty$.

   Claim: f is not strongly continuous

   We have $\|f(\frac{1}{n})e_k\| = \|e_{n+k}\| = 1$ which does not go to zero for $n \to \infty$.

3. That $(T_\phi(f), T_\phi(g)) = (f,g)$ follows from translation invariance of the integral.

   Claim: T is not norm continuous.

   Let $\alpha \in [-\pi,\pi]$, $\alpha \neq 0$ be given. We have $\|T_\alpha - id\| \geq \|T_\alpha f - f\|$ for any choice $f \in L^2(U(1))$ with $\|f\| = 1$. Pick a continuous function f with support in $[-\alpha/2, \alpha/2]$ and $\|f\| = 1$. Note that f and $T_\alpha f$ are orthogonal as they have disjoint support. Thus

   $$\|T_\alpha f - f\|^2 = \|T_\alpha f\|^2 + \|f\|^2 = 2 \ ,$$

   where we used that T_α is unitary. We already saw that $\|T_\alpha - id\| \leq 2$ so that we in fact have

   $$\|T_\alpha - id\| = 2 \quad \text{for all } \alpha \in [-\pi,\pi],\ \alpha \neq 0 \ .$$

   This does not approach zero as $\alpha \to 0$.

   Let us also check that T is strongly continuous.

   Note that T is a group homomorphism. Thus $\|T_\alpha f - T_\beta f\| \leq \|T_\beta\|\,\|T_{\alpha-\beta} f - f\|$. Thus to see that for $\alpha \to \beta$ we have $\|T_\alpha f - T_\beta f\| \to 0$ it is enough to check that for all $\alpha \to 0$ we have $\|T_\alpha f - f\| \to 0$.

   Claim: If $\|T_\alpha e_n - e_n\| \to 0$ for $\alpha \to 0$ for all elements of an ON-basis $\{e_n\}_{n\in\mathbb{N}}$ then T_n is strongly continuous.

   Let $f = \sum_{n=1}^\infty f_n e_n$ and $\varepsilon > 0$ be given. Split $f = x + y$ where $x = \sum_{n=0}^N f_n e_n$ and $y = \sum_{n=N+1}^\infty f_n e_n$. By choosing N large enough we can achieve $\|y\| < \varepsilon$. Pick δ small enough such that for $|\varphi| < \delta$ we have $\|T_\varphi f_n e_n - f_n e_n\| < \varepsilon/N$ for all $n = 1, \dots, N$. Then

   $$\|(T_\varphi - id)f\| \leq \sum_{n=1}^N \|(T_\varphi - id) f_n e_n\| + \|(T_\varphi - id)y\| \leq N\varepsilon/N + \|(T_\varphi - id)\|\varepsilon$$

But T_φ is unitary and so $\|T_\varphi - id\| \le \|T_\varphi\| + \|id\| = 2$. Hence for all $|\varphi| < \delta$ we have $\|(T_\varphi - id)f\| < 3\varepsilon$.

Consider the ON-basis $e_n(\psi) = e^{in\psi}$ as in Exercise 29. On this ON-basis one quickly checks that $\|T_\alpha e_n - e_n\| \to 0$ for $\alpha \to 0$ does indeed hold.

Exercise 29

1. $(R_m(\sum \lambda_n e_n), R_m(\sum \mu_n e_n)) = \sum \bar{\lambda}_n \mu_n = (\sum \lambda_n e_n, \sum \mu_n e_n)$.

2. Using the hint we see that R_m maps smooth functions to smooth functions. So does T_φ. Thus the linear subspace of all smooth functions in $L^2(U(1))$ is invariant. (But it is not closed.)

3. Let $v \in \mathcal{H}$ be a non-zero vector. Write $\mathcal{L} \subset \mathcal{H}$ for the closure of span$(G.v)$. Then $\mathcal{L}$ is a Hilbert space containing $T_\varphi(v)$ for all φ. The function $f : [0, 2\pi] \to \mathcal{H}$, $\varphi \mapsto T_\varphi(v)$ is continuous (since by Exercise 28 part 3 the map T is strongly continuous). Let $\xi := \int_0^{2\pi} f(\varphi)d\varphi$. As the integral can be approximated by Riemann sums, each of which lies in span$(G.v)$, we have $\xi \in \mathcal{L}$. Now by definition of the integral,

$$(e_m, \xi) = \int_0^{2\pi} (e_m, T_\varphi(v))d\varphi = \int_0^{2\pi} (T_{-\varphi}(e_m), v)d\varphi = \int_0^{2\pi} e^{im\varphi}(e_m, v)d\varphi$$
$$= 2\pi\delta_{m,0}(e_0, v) \ .$$

   Thus $\xi = 2\pi(e_0, v)e_0$ and so if $(e_0, v) \neq 0$, then also $e_0 \in \mathcal{L}$. Applying this to $R_m v$ shows that all e_m for which $(e_m, v) \neq 0$ are contained in $\mathcal{L}$. Since $v \neq 0$, there is at least one m such that $(e_m, v) \neq 0$. But if $\mathcal{L}$ contains one e_m, then it contains all as we can shift via R_k. Hence $\mathcal{L} = \mathcal{H}$.

4. Consider the subspace $\mathcal{L} = L^2([0, \pi]) \subset L^2(U(1))$ of all functions which are identically zero on half of the unit circle. By the hint in part 2, R_m maps $\mathcal{L}$ to itself. Thus $\{0\} \subsetneq \mathcal{L} \subsetneq \mathcal{H}$ is a closed invariant subspace.

   However, there are no one-dimensional invariant subspaces. Indeed, suppose a non-zero $v \in \mathcal{H}$ satisfies $R_m v = \mu_m v$ for all $m \in \mathbb{Z}$ and some $\mu_m \in \mathbb{C}$, $|\mu_m| = 1$. Evaluating this on φ gives

$$e^{im\varphi}v(\varphi) = \mu_m v(\varphi) \ ,$$

   for all $\varphi \in \mathbb{R}$, which is impossible for $m \neq 0$.