Solutions for exercise sheet # 06
Topics in representation theory WS 2017

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Exercise 22

1. We have, for an appropriate $\lambda \in k^\times$,

$$\bar{\rho}(g)\bar{\rho}(h) = \pi(\rho(g)\rho(h)) = \pi(\lambda\rho(gh)) = \pi(\lambda id)\pi(\rho(gh)) = \bar{\rho}(gh)) \ .$$

2. "$\Rightarrow$": By assumption, there is a linear isomorphism $f : W \to W'$ such that for all $g \in G$, $f \circ \rho(g) = \lambda_g \cdot \rho'(g) \circ f$ for some constants $\lambda_g \in k^\times$. Thus

$$\begin{aligned} \bar{\rho}'(g) &= \pi(\rho'(g)) = \pi(\lambda_g^{-1} f\rho(g)f^{-1}) = \pi(\lambda_g^{-1} id)\pi(f\rho(g)f^{-1}) \\ &= \overline{f(\ )f^{-1}} \circ \pi(\rho(g)) \ . \end{aligned}$$

"$\Leftarrow$": Similar as above: From $\bar{\rho}'(g) = \overline{f(\ )f^{-1}} \circ \bar{\rho}(g)$ one concludes the existence of $\lambda_g \in k^\times$ such that $\lambda_g\rho'(g) = f \circ \rho(g) \circ f^{-1}$.

3. Since $\pi(\rho(g)\rho(h)\rho(gh)^{-1}) = \pi(\rho(g))\pi(\rho(h))\pi(\rho(gh))^{-1} = \sigma(g)\sigma(h)\sigma(gh)^{-1} = 1$, we conclude that $\rho(g)\rho(h)\rho(gh)^{-1} = \lambda id_W$ for some $\lambda \in k^\times$, that is, $\rho(g)\rho(h) \in k^\times \cdot \rho(gh)$.

   Analogously, for another choice ρ' we have $\pi(\rho'(g)\rho(g)^{-1}) = \pi(\rho'(g))\pi(\rho(g))^{-1} = \sigma(g)\sigma(g)^{-1} = 1$, hence $\rho'(g) = \mu_g \cdot \rho(g)$ for some constants $\mu_g \in k^\times$. This shows that the identity map $W \to W$ provides an isomorphism of projective representations $(W, \rho) \to (W, \rho')$.

Exercise 23

f exists: Recall that $\widehat{G} = \{(F, g) \in GL(W) \times G | \pi(F) = \bar{\rho}(g)\}$. Define

$$f(h) := (b(h), a(h)) \ .$$

This is indeed an element of $\widehat{G}$ by the first commuting diagram in the exercise that H is assumed to satisfy. It also makes the second diagram in the exercise commute: $p(f(h)) = a(h)$ and $\hat{\rho}(f(h)) = b(h)$ by construction, as p and $\hat{\rho}$ just project to the first and second entry, respectively.

f is unique: Suppose another group homomorphism f' makes the second diagram commute. Then we can write $f'(h) = (u(h), v(h))$. But $u(h) = \hat{\rho}(f'(h)) = b(h)$, etc., and so $f' = f$.

Exercise 24

1. (a) The condition $\delta\chi = 1$ reads

$$\chi(h,k)\,\chi(gh,k)^{-1}\chi(g,hk)\,\chi(g,h)^{-1} = 1\ .$$

Setting $h = e$ in the above expression gives

$$\chi(e,k)\chi(g,e)^{-1} = 1\ .$$

Further setting $k = e$ or setting $g = e$ gives the first result.
For the second result set $h = g^{-1}$ and $k = g$ in the condition $\delta\chi = 1$ to get

$$\chi(g^{-1},g)\,\chi(e,g)^{-1}\chi(g,e)\,\chi(g,g^{-1})^{-1} = 1\ .$$

As we just checked, the two middle terms cancel.

(b) We make the ansatz $\beta(g) = \chi(g,g)^{-1}$. Then

$$(\delta\beta)(e,g) = \beta(g)\beta(g)^{-1}\beta(e) = \chi(e,e)^{-1}\ ,$$

and in the same way $(\delta\beta)(g,e) = \beta(e)$. Define $\chi' = \chi\delta\beta$. Then

$$\chi'(e,g) = \chi(e,g)(\delta\beta)(g,e) = \chi(e,g)\chi(e,e)^{-1} = 1$$

by part a. Again by part a, this implies $\chi'(g,e) = 1$ as well.

2.
   - *(unit)* The unit is $(\chi(e,e)^{-1},e)$. For example,

     $$(a,g)(\chi(e,e)^{-1},e) = (a\chi(e,e)^{-1}\chi(g,e),g) = (a,g)\ ,$$

     where in the last equality we used part 1a.
   - *(inverse)* The inverse of (a,g) is $(a^{-1}\chi(g,g^{-1})^{-1},g^{-1})$. To check that this is a 2-sided inverse one again needs to use part 1a.
   - *(central extension)* We need to check that $j(a) := (a\chi(e,e)^{-1},e)$ is a group homomorphism, the rest is clear. We have

     $$j(a)j(b) = (a\chi(e,e)^{-1},e)(b\chi(e,e)^{-1},e) = (ab\chi(e,e)^{-2}\chi(e,e),e) = j(ab)\ .$$

Exercise 25

1. Write $h = p(g)$. Since $g^2 \in j(A)$ we have $e = p(g^2) = h^2$. Suppose there is a group homomorphism $s : H \to G$ such that $p(s(h)) = h$ for all $h \in H$.

   We have $p(s(h)g^{-1}) = p(s(h))p(g)^{-1} = hh^{-1} = e$, and so $s(h)g^{-1} = j(a)$ for some $a \in A$. Or, equivalently, $s(h) = j(a)g$. Thus

   $$e = s(e) = s(h^2) = s(h)s(h) = j(a)^2g^2\ .$$

   But then $g^2 = j(a^{-1})^2$, in contradiction to the assumption that g^2 cannot be written as a square in A.

2. No element in $A = \{\pm 1\}$ squares to -1. But the element $g = \mathrm{diag}(i, -i) \in SU(2)$ satisfies $g^2 = -id$.

3. There exists a unique group homomorphism $\tilde{\iota} : PSU(2) \to PU(2)$ making the diagram

$$\begin{array}{ccc} SU(2) & \longrightarrow & PSU(2) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tilde{\iota}} \\ U(2) & \longrightarrow & PU(2) \end{array}$$

   commute. In fact, $\tilde{\iota}$ is a bijection: it is surjective (the composition $SU(2) \to U(2) \to PU(2)$ is surjective) and injective (if $\tilde{\iota}(\pi(x)) = 0$, then $x = \lambda id$, i.e. $\pi(x) = 1$).

   Suppose there is a splitting homomorphism $s : PU(2) \to U(2)$.

   Claim: For all $x \in PU(2)$, $\det(s(x)) = 1$.

   Proof: We get a group homomorphism

$$SU(2) \xrightarrow{\pi} PSU(2) \xrightarrow{\tilde{\iota}} PU(2) \xrightarrow{s} U(2) \xrightarrow{\det} U(1) \ .$$

   As $SU(2)$ is generated by commutators and $U(1)$ is commutative, all elements of $SU(2)$ must get mapped to 1. But $SU(2) \xrightarrow{\pi} PSU(2) \xrightarrow{\tilde{\iota}} PU(2)$ is surjective, and so $\det(s(x)) = 1$ must hold for all x, proving the claim.

   The claim shows that the image of s is actually in $SU(2)$. Combining with $\tilde{\iota}$, we obtain a group homomorphism $s' : PSU(2) \to SU(2)$ such that $\pi \circ s' = id_{PSU(2)}$. But this would be a splitting homomorphism for

$$\{\pm 1\} \longrightarrow SU(2) \longrightarrow PSU(2) \ ,$$

   which by part 2 does not exist.

4. Let us go through the argument of parts 1–3 backwards. Suppse $U(1) \to U(N) \to PU(N)$ splits via a group homomorphism $s : PU(N) \to U(N)$. Then we get a group homomorphism

$$SU(N) \xrightarrow{\pi} PSU(N) \xrightarrow{\sim} PU(N) \xrightarrow{s} U(N) \xrightarrow{\det} U(1) \ .$$

   $SU(N)$ (and every semisimple compact Lie group for that matter) is generated by commutators, and so again its image is 1. We are reduced to looking at the central extension

$$\mathbb{Z}_N \xrightarrow{x \mapsto e^{2\pi i x/N} id} SU(N) \xrightarrow{\pi} PSU(N) \ .$$

   Now one generalises the argument in part 1 from "square" to "N'th power" to see that this sequence does not split.

The central extension $U(1) \to UA(\mathcal{H}) \xrightarrow{\gamma} \operatorname{Aut}\mathbb{P}(\mathcal{H})$ also does not split. Indeed, let $U(\mathbb{P}(\mathcal{H})) := \gamma(U(\mathcal{H}))$ be the image of all unitary maps. This is a subgroup, and we have an exact sequence (and central extension)

$$U(1) \to U(\mathcal{H}) \xrightarrow{\gamma} U(\mathbb{P}(\mathcal{H})) \ .$$

The fact that this is an exact sequence implies in particular that $U(\mathbb{P}(\mathcal{H})) \cong U(\mathcal{H})/U(1) = PU(\mathcal{H})$.

Since a splitting of $U(1) \to UA(\mathcal{H}) \to \operatorname{Aut}\mathbb{P}(\mathcal{H})$ also gives a splitting of $U(1) \to U(\mathcal{H}) \xrightarrow{\gamma} U(\mathbb{P}(\mathcal{H}))$, we see that for $\mathcal{H} = \mathbb{C}^N$, $N \geq 2$, this central extension does not split.

The fact that $U(1) \to UA(\mathcal{H}) \xrightarrow{\gamma} \operatorname{Aut}\mathbb{P}(\mathcal{H})$ is a central extensions implies in particular that $UA(\mathcal{H})/U(1) \cong \operatorname{Aut}\mathbb{P}(\mathcal{H})$.