Solutions for exercise sheet # 05
Topics in representation theory WS 2017

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Exercise 18

1. The only thing we need to show is that for any choice of e', f' we have $(e', f') = 0$. But this is immediate from
$$\delta([e'], [f']) = \delta(T[e], T[f]) = \delta([e], [f]) = 0 \ .$$

2. Write $x = \lambda e + \mu f$. Choose $x' \in \mathcal{H}$ s.th. $[x'] = T[x]$ and $\|x\| = \|x'\|$. Then
$$\begin{aligned}\|x' - (e', x')e' - (f', x')f'\|^2 &= \cdots = \|x'\|^2 - 2|(e', x')|^2 - 2|(f', x')|^2 \\ &= \|x'\|^2 - 2\|x'\|^2\delta([e'], [x']) - 2\|x'\|^2\delta([f'], [x']) \\ &= \|x\|^2 - 2\|x\|^2\delta([e], [x]) - 2\|x\|^2\delta([f], [x]) \\ &= \cdots = \|x - (e, x)e - (f, x)f\|^2 = 0 \ .\end{aligned}$$
This shows that $x' = \lambda' e' + \mu' f'$ with $\lambda' = (e', x')$ and $\mu' = (f', x')$.

Exercise 19
Let $v = \lambda e + \tilde{v}$ and $w = \mu e + \tilde{w}$ with $\tilde{v}, \tilde{w} \in \mathcal{P}$ and let $z \in \mathbb{C}$. Then
$$\begin{aligned}X(v + w) &= X((\lambda + \mu)e + \tilde{v} + \tilde{w}) = \chi(\lambda + \mu)e' + V(\tilde{v} + \tilde{w}) \\ &= \chi(\lambda)e' + \chi(\mu)e' + V(\tilde{v}) + V(\tilde{w}) = X(v) + X(w) \ .\end{aligned}$$
Similarly:
$$X(zv) = \chi(z)X(v) \ .$$
Thus X is (anti-)linear.
To see compatibility with the scalar product compute
$$\begin{aligned}(Xv, Xw) &= (\chi(\lambda)e' + V(\tilde{v}), \chi(\mu)e' + V(\tilde{w})) = \overline{\chi(\lambda)}\chi(\mu) + (V(\tilde{v}), V(\tilde{w}))) \\ &= \chi(\bar{\lambda}\mu + (\tilde{v}, \tilde{w})) = \chi((\lambda e + \tilde{v}, \mu e + \tilde{w})) = \chi((v, w)) \ ,\end{aligned}$$
where we used in particular that $(V(\tilde{v}), V(\tilde{w})) = \chi((\tilde{v}, \tilde{w}))$.
Finally, for $v \neq 0$, X satisfies
$$[Xv] = [\chi(\lambda)e' + V(\tilde{v})] = (*)$$
If $\lambda = 0$ we use Claim 1, namely that $(*) = [V(\tilde{v})] = T[\tilde{v}]$ to get $[Xv] = T[v]$. If $\lambda \neq 0$ we continue with
$$(*) = [e' + \chi(\lambda^{-1})V(\tilde{v})] = [e' + V(\lambda^{-1}\tilde{v})] = T[e + \lambda^{-1}\tilde{v}] = T[\lambda e + \tilde{v}]$$
where in the next to last step we substituted the definition of V.

Exercise 20

1. Applying γ to $f := \tilde{\rho}(g)\tilde{\rho}(h)\tilde{\rho}(gh)^{-1}$ and using that γ is a group homomorphism together with the commuting diagram gives:

$$\gamma(f) = \gamma(\tilde{\rho}(g))\gamma(\tilde{\rho}(h))\gamma(\tilde{\rho}(gh))^{-1} = \rho(g)\rho(h)\rho(gh)^{-1} = id$$

where in the last step we used that ρ is a group homomorphism. Since $\dim \mathcal{H} \geq 2$, $\ker \gamma = U(1)$ and so

$$\tilde{\rho}(g)\tilde{\rho}(h)\tilde{\rho}(gh)^{-1} = \chi(g,h)id$$

for some (unique) $\chi(g,h) \in U(1)$.

2. This follows form associativity of the composition of maps:

$$\begin{aligned} & \tilde{\rho}(g)(\tilde{\rho}(h)\tilde{\rho}(k)) = (\tilde{\rho}(g)\tilde{\rho}(h))\tilde{\rho}(k) \\ \Leftrightarrow\quad & \tilde{\rho}(g)\chi(h,k)\tilde{\rho}(hk) = \chi(g,h)\tilde{\rho}(gh)\tilde{\rho}(k) \\ \Leftrightarrow\quad & \alpha_g(\chi(h,k))\tilde{\rho}(g)\tilde{\rho}(hk) = \chi(g,h)\tilde{\rho}(gh)\tilde{\rho}(k) \\ \Leftrightarrow\quad & \alpha_g(\chi(h,k))\chi(g,hk)\tilde{\rho}(g(hk)) = \chi(g,h)\chi(gh,k)\tilde{\rho}((gh)k) \end{aligned}$$

The result now follows from associativity of the composition in G.

3. By the same reason as above, for all g we have $\tilde{\rho}'(g)\tilde{\rho}(g)^{-1} = \beta(g)id$ for some $\beta(g) \in U(1)$. From this one computes

$$\begin{aligned} & \tilde{\rho}'(g)\tilde{\rho}'(h) = \chi'(g,h)\tilde{\rho}'(gh) \\ \Leftrightarrow\quad & \beta(g)\tilde{\rho}(g)\beta(h)\tilde{\rho}(h) = \beta(gh)\tilde{\rho}(gh) \\ \Leftrightarrow\quad & \beta(g)\alpha_g(\beta(h))\tilde{\rho}(g)\tilde{\rho}(h) = \beta(gh)\tilde{\rho}(gh) \\ \Leftrightarrow\quad & \beta(g)\alpha_g(\beta(h))\chi(g,h)\tilde{\rho}(gh) = \beta(gh)\tilde{\rho}(gh)\ . \end{aligned}$$

Exercise 21

1. Let $\lambda \in \mathbb{C}$. The line through $(1,0)$ and $(0,\lambda)$ is parametrised by $l(t) = (1-t)(1,0)+t(0,\lambda)$. This line intersects the unit sphere whenever $|l(t)|^2 = 1$, i.e. when

$$|1-t|^2 + |t\lambda|^2 = 1 \quad \Leftrightarrow \quad t(-2+t+t|\lambda|^2) = 0\ .$$

One of the intersection points is $(1,0)$ at $t = 0$ and the other is at $t = 2/(1+|\lambda|^2)$. Altogether,

$$l(\tfrac{2}{1+|\lambda|^2}) = \tfrac{-1+|\lambda|^2}{1+|\lambda|^2}(1,0) + \tfrac{2}{1+|\lambda|^2}(0,\lambda)\ .$$

From the construction it is clear that this map is injective and surjective.

To complete the maps given in the exercise to a map from all of S^2 to all of $\mathbb{P}(\mathbb{C}^2)$ we declare that the point $(1,0) \in S^2$ gets mapped to $\mathbb{C}e_2 \in \mathbb{P}(\mathbb{C}^2)$.

Denote the overall isomorphism $\mathbb{P}(\mathbb{C}^2) \to S^2$ by φ.

For later use we note that the Euclidean scalar product on $\mathbb{R}^3$ under the identification with $\mathbb{R} \times \mathbb{C}$ takes the form

$$\big((x,z),(y,w)\big) = xy + \tfrac{1}{2}(\bar{z}w + z\bar{w}) \ .$$

Let $\alpha, \beta \in \mathbb{P}(\mathbb{C}^2)$. For the relation between d_{S^2} and d_{FS} we distinguish three cases

- $\alpha = \beta = \mathbb{C}e_2$: Both distances are zero.
- $\alpha = \mathbb{C}e_2$, $\beta \neq \mathbb{C}e_2$: The image points are $\varphi(\alpha) = (1,0)$ and, for $\beta = [e_1 + \lambda e_2]$,
$$\varphi(\beta) = \frac{1}{|\lambda|^2+1}\big(|\lambda|^2 - 1\,,\, 2\lambda\big)$$
Now $\delta(\alpha,\beta) = |(e_2, e_1 + \lambda e_2)|^2/(1+|\lambda|^2) = |\lambda|^2/(1+|\lambda|^2)$. By the hint we get
$$\cos d_{FS}(\alpha,\beta) = 2\delta(\alpha,\beta) - 1 = \frac{|\lambda|^2-1}{1+|\lambda|^2} \ .$$
But this is the same as the inner product $((1,0),\varphi(\beta))$ of unit vectors in $\mathbb{R}^3$, i.e. as the cosine of the angle between these unit vectors, or the cosine of the geodesic distance between the points:
$$\cos d_{S^2}(\varphi(\alpha),\varphi(\beta)) = \big(\varphi(\alpha),\varphi(\beta)\big) \ .$$
- $\alpha, \beta \neq \mathbb{C}e_2$: Write $\alpha = [e_1 + \lambda e_2]$, $\beta = [e_1 + \mu e_2]$. Then
$$\delta(\alpha,\beta) = \frac{|(e_1+\lambda e_2, e_1 + \mu e_2)|^2}{(1+|\lambda|^2)(1+|\mu|^2)} = \frac{|1+\bar{\lambda}\mu|^2}{(1+|\lambda|^2)(1+|\mu|^2)}$$
and
$$\begin{aligned}\cos d_{FS}(\alpha,\beta) &= \frac{2 + 2\bar{\lambda}\mu + 2\lambda\bar{\mu} + 2|\lambda|^2|\mu|^2 - 1 - |\lambda|^2 - |\mu|^2 - |\lambda|^2|\mu|^2}{(1+|\lambda|^2)(1+|\mu|^2)} \\ &= \frac{1 + 2\bar{\lambda}\mu + 2\lambda\bar{\mu} + |\lambda|^2|\mu|^2 - |\lambda|^2 - |\mu|^2}{(1+|\lambda|^2)(1+|\mu|^2)} \ .\end{aligned}$$
On the other hand
$$(\varphi(\alpha),\varphi(\beta)) = \frac{(|\lambda|^2-1)(|\mu|^2-1) + \frac{1}{2}(2\bar{\lambda}2\mu + 2\lambda 2\bar{\mu})}{(1+|\lambda|^2)(1+|\mu|^2)} \ ,$$
which is the same.

2. A metric has three properties: 1) it takes values in $\mathbb{R}_{\geq 0}$ and distance 0 implies the arguments are equal; 2) it is symmetric; 3) it satisfies the triangle inequality: for all points a, b, c,

$$d(a,c) \leq d(a,b) + d(b,c) \ .$$

We abbreviate $d := d_{FS(\mathcal{H})}$ for the Fubini Studi function on $\mathcal{H}$, which we want to establish to be a metric. Only the third property is not obvious.

For $\dim \mathcal{H} = 1$ there is nothing to do ($\mathbb{P}\mathcal{H}$ is a single point). Assume now $\dim \mathcal{H} \geq 2$.

Let $\alpha, \beta, \gamma \in \mathbb{P}(\mathcal{H})$ be arbitrary and choose $a, b, c \in \mathcal{H}$ such that $\alpha = [a]$, etc. We distinguish two cases:

- Suppse a, b, c are linearly dependent. Then the lie in some two-dimensional subspace $\mathcal{E} \subset \mathcal{H}$. Choose an embedding $\psi : \mathbb{C}^2 \to \mathcal{H}$ with image $\mathcal{E}$, compatible with the inner product. Let $a', b', c' \in \mathbb{C}^2$ satisfy $a = \psi(a')$, etc. Then
$$\delta_{\mathcal{H}}(\alpha, \beta) = \delta_{\mathcal{H}}([\psi(a')], [\psi(b')]) = \delta_{\mathbb{C}^2}([a'], [b']) \ ,$$
and similar for all other pairs. Thus the triangle inequality for $\alpha, \beta, \gamma \in \mathbb{P}(\mathcal{H})$ reduces to that of $[a'], [b'], [c'] \in \mathbb{P}(\mathbb{C}^2)$, which we proved in part 1.
- Suppse a, b, c are linearly independent. Let $\mathcal{E}$ be the two-dimensional subspace of $\mathcal{H}$ spanned by a and c. Write $b' \in \mathcal{E}$ for the image of b under orthogonal projection $\mathcal{H} \to \mathcal{E}$. In formulas, for an ON basis e_1, e_2 of $\mathcal{E}$,
$$b' = (e_1, b)e_1 - (e_2, b)e_2 \ .$$
Note that $b - b'$ is orthogonal to $\mathcal{E}$.
Suppose $b' = 0$. Then b is orthogonal to a and b and one computes $\delta([a],[b]) = 0 = \delta([b],[c])$. Thus $d_{FB(\mathcal{H})}([a],[b]) = \pi = d_{FB(\mathcal{H})}([c],[b])$, and $d_{FB(\mathcal{H})}([a],[c]) \leq d_{FB(\mathcal{H})}([a],[b]) + d_{FB(\mathcal{H})}([b],[c]) = 2\pi$ is trivially true.
Suppose that $b' \neq 0$. It then makes sense to consider $[b']$. From the previous case we know that
$$d_{FB(\mathcal{H})}([a],[c]) \leq d_{FB(\mathcal{H})}([a],[b']) + d_{FB(\mathcal{H})}([b'],[c]) \ .$$
The general result follows from the observation that
$$\left(\cos \tfrac{d_{FB(\mathcal{H})}([a],[b'])}{2}\right)^2 = \delta_{\mathcal{H}}([a],[b']) = \left| \frac{(a,b')}{\|a\| \, \|b'\|} \right|^2 \overset{(*)}{=} \left| \frac{(a,b)}{\|a\| \, \|b'\|} \right|^2$$
$$\overset{(**)}{\geq} \left| \frac{(a,b)}{\|a\| \, \|b\|} \right|^2 = \delta_{\mathcal{H}}([a],[b]) = \left(\cos \tfrac{d_{FB(\mathcal{H})}([a],[b])}{2}\right)^2 \ .$$

Here $(*)$ follows as $b - b'$ is orthogonal to a by construction, and $(**)$ follows from $\|b'| \leq \|b\|$. Since cos is monotonically decreasing in the relevant range $d \in [0, \pi]$, we get

$$d_{FB(\mathcal{H})}([a], [b']) \leq d_{FB(\mathcal{H})}([a], [b]) \ .$$

Thus

$$d_{FB(\mathcal{H})}([a], [b']) + d_{FB(\mathcal{H})}([b'], [c]) \leq d_{FB(\mathcal{H})}([a], [b]) + d_{FB(\mathcal{H})}([b], [c]) \ ,$$

completing the proof of the triangle inequality.