Solutions for exercise sheet # 04
Topics in representation theory WS 2017

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Exercise 13

The topological definition of "connected" is: If $G = U \cup V$ is a disjoint union with both U and V open, then either U or V is empty. Denote the set in the statement of the problem by H and write $K = G \setminus H$. The aim is to show that both H and K are open. Since $e \in H$, H is not empty and so $K = \emptyset$.
H is clearly open (a point $h \in H$ has open neighbourhood $h \cdot V$). K is open for the same reason: if $k \in K$ then also $k \cdot V \subset K$, for if not we can write $k \cdot x = g_1 \cdots g_n$ with $x, g_1, \dots, g_n \in V$, and so $k \in G$, contradiction.

Exercise 14

It is easy to see that $f(t+\varepsilon) = f(t)f(\varepsilon)$ and $g(t+\varepsilon) = g(t)g(\varepsilon)$. Since matrix multiplication is bilinear we therefore have

$$f'(t_0) = \tfrac{d}{d\varepsilon} f(t)f(\varepsilon)|_{\varepsilon=0} = f(t)\tfrac{d}{d\varepsilon} f(\varepsilon)|_{\varepsilon=0} = f(t_0)f'(0) \ .$$

Similarly, for g we find $g'(t_0) = g(t_0)g'(0)$. Next we compute the derivatives at $t = 0$:

$$f'(0) = \tfrac{d}{dt}\rho(\exp(tX))|_{t=0} = D\rho(\tfrac{d}{dt}\exp(tX)|_{t=0}) = D\rho(X) \ ,$$

and clearly also $g'(0) = D\rho(X)$. Thus both f and g solve $h'(t) = h(t)D\rho(X)$ with initial condition $f(0) = id_W = g(0)$. Therefore $f(t) = g(t)$ for all t, in particular for $t = 1$.

If one applies this to the one-dimensional representation of $GL(N, \mathbb{C})$ given by $\rho(M) = \det(M)$ and uses $\det(1 + tX) = 1 + t\,\mathrm{tr}(X) + O(t^2)$ to compute $D\rho(X) = \mathrm{tr}(X)$ one finds the claimed special case.

Exercise 15

Semidirect product: Let N, H be groups and $\varphi : H \to \mathrm{Aut}(N)$ a group homomorphism. Then $N \rtimes_\varphi H$ is $N \times H$ as a set and has the product

$$(n, h)(n', h') = (n\varphi_h(n'), hh') \ .$$

Let $A \in UA(\mathcal{H})$ be any choice of anti-unitary map (these always exist, e.g. take any ON-basis of $\mathcal{H}$ and define A by complex conjugating the coefficients).
Consider $U(\mathcal{H}) \rtimes_\varphi \mathbb{Z}_2$ where $\varphi : \mathbb{Z}_2 \to U(\mathcal{H})$ maps the generator $1 \in \mathbb{Z}_2$ to $A(-)A^{-1} \in \mathrm{Aut}(U(\mathcal{H}))$. We claim that

$$F : U(\mathcal{H}) \rtimes_\varphi \mathbb{Z}_2 \to UA(\mathcal{H}) \quad , \quad (U, a) \mapsto UA^a$$

is a group-isomorphism. Injectivity is clear, for surjectivity note that if X is unitary, then $F(X,0) = X$; if X is anti-unitary then $F(XA^{-1},1) = XA^{-1}A = X$. It remains to check that F is a group-homomorphism:

$$\begin{aligned} F((U,a)(U',a')) &= F(UA^aU'A^{-a}, a+a') = UA^aU'A^{-a}A^{a+a'} \\ &= UA^aU'A^{a'} = F(U,a)F(U',a') \ . \end{aligned}$$

Exercise 16

For another representative $\varphi' = \lambda\varphi$ we have

$$T([\varphi']) = [X\varphi'] = [X(\lambda\varphi)] = [\tilde{\lambda}X(\varphi)] = [X\varphi] = T([\varphi]) \ ,$$

where $\tilde{\lambda} = \lambda$ if X is unitary and $\tilde{\lambda} = \bar{\lambda}$ if X is anti-unitary.
To check that $T \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ we compute, for X unitary,

$$\delta(T[\varphi], T[\psi]) = \delta([X\varphi],[X\psi]) = \Big|\frac{(X\varphi, X\psi)}{||\varphi||\,||\psi||}\Big|^2 = \Big|\frac{(\varphi,\psi)}{||\varphi||\,||\psi||}\Big|^2 = \delta([\varphi],[\psi]) \ .$$

For X anti-unitary, the calculation is the same except that $(X\varphi, X\psi) = (\psi,\varphi)$, which makes no difference due to the absolute value.
That T is invertible follows form the next computation and from $T = id$ for $X = id$.
The group-homomorphism property follows from

$$\gamma(X)\gamma(Y)([\varphi]) = \gamma(X)([Y\varphi]) = [XY\varphi] = \gamma(XY)([\varphi]) \ .$$

Exercise 17

1. (sketch) Let $\Lambda \in L_+^\uparrow$ be given. Write $v = (\Lambda_{10}, \Lambda_{20}, \Lambda_{30})$, i.e. the first column of Λ less its first entry.

   If $v = 0$ one can verify (using $\Lambda^t J\Lambda = J$ and $\Lambda_{00} \geq 1$, $\det(\Lambda) = 1$) that in this case $\Lambda = \hat{R}$ for some $R \in SO(3)$, so we are done. E.g. to see $\Lambda_{01} = 0$ recall that the condition $\Lambda^t J\Lambda = J$ just encodes the property $\eta(\Lambda x, \Lambda y) = \eta(x,y)$ for all $x, y \in \mathbb{R}^4$. Now take $x = e_0$, $y = e_0$ to see that $(\Lambda_{00})^2 = 1$ and $x = e_0$, $y = e_1$ to get $\Lambda_{01} = 0$.

   Assume now that $v \neq 0$. Pick a rotation $R \in SO(3)$ such $Rv = (r,0,0)$. Write $\hat{R}$ for the block-diagonal matrix with 1 in the upper left corner and R in the remaining 3×3 block. Then

$$\hat{R}\Lambda = \begin{pmatrix} * & * & * & * \\ r & * & * & * \\ 0 & y_1 & y_2 & y_3 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix} \ .$$

Denote the two 3-vectors by $\vec{y}$ and $\vec{z}$. Note that $\vec{y}$ and $\vec{z}$ are orthonormal (since $R\Lambda \in L$). Pick the unique $\vec{x}$ such that the 3×3 matrix R' with columns $(\vec{x}, \vec{y}, \vec{z})$ is in $SO(3)$ (that is, take $\vec{x} = \vec{y} \times \vec{z}$). Then

$$\hat{R}\Lambda\hat{R}' = \begin{pmatrix} a & c & * & * \\ b & d & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

But $M := \hat{R}\Lambda\hat{R}' \in L$ and so M satisfies $M^t J M = J$ where $J = \mathrm{diag}(1, -1, -1, -1)$. With some fiddling (similar to what we did in the case $v = 0$) one finds that all "$*$" entries are actually zero, so that

$$\hat{R}\Lambda\hat{R}' = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

One can write down all solutions to a, b, c, d such that $M^t J M = J$ and $M_{00} \geq 1$, $\det(M) = 1$, and a short calculation gives

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. Thus $\hat{R}\Lambda\hat{R}'$ is a boost, as required.

Connectedness is now clear.

2. (sketch) Define

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

Then Id, P, T, PT lie in different connected components of L as discussed in Remark 2.1.7.

The rest of the computation is very similar to the $UA(\mathcal{H})$ computation above. Namely after appropriate choice of φ one checks that

$$\psi : L_+^\uparrow \rtimes_\varphi (\mathbb{Z}_2 \times \mathbb{Z}_2) \to L \quad , \quad (\Lambda, a, b) \mapsto \Lambda P^a T^b$$

is an isomorhism.