Solutions for exercise sheet # 03
Topics in representation theory WS 2017

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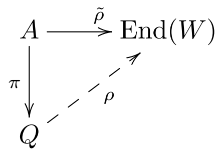
Exercise 9

1. Each Q-representation $\rho : Q \to \mathrm{End}(W)$ defines an A-representation via $\rho \circ \pi : A \to \mathrm{End}(W)$.

   An A-representation $\tilde{\rho} : A \to \mathrm{End}(W)$ defines a Q-representation if and only if it vanishes on the kernel of π. In this case there is a unique ρ such that the diagram

   commutes.

   Let (ρ_M, M) and (ρ_N, N) be Q-representations. Denote the correponding A-representations by $\tilde{M}, \tilde{N}$. Then $\mathrm{Hom}_Q(M, N) = \mathrm{Hom}_A(\tilde{M}, \tilde{N})$ (they are the same subspace of the space of k-linear maps $M \to N$).

   This shows that two distinct simple Q-representations are also distinct simple A-representations. However, in general not every simple A-representation descends to a Q-representation. Thus #(simple A-modules) $\geq$ #(simple Q-modules).

2. Let Q be a quotient of A, i.e. there is a surjective algebra homomorphism $\pi : A \to Q$. Suppose A is semisimple.

   Let M be a Q-module. We need to show that $M = \bigoplus_i M_i$ for some family of simple Q-modules M_i. As in part 1, let $\tilde{M}$ be the corresponding A-module. As A is semisimple, there are simple A-modules W_j such that $\tilde{M} = \bigoplus_j W_j$. But by definition, the action $\tilde{\rho} = \rho \circ \pi : A \to \mathrm{End}(M)$ vanishes on $\ker(\pi)$. Hence so does the action of A on all W_j. By part 1, we obtain a simple Q-module which we will also call W_j and a decomposition $M = \bigoplus_j W_j$ of Q-modules.

   Thus every Q-module is a direct sum of simple ones, hence semisimple.

Exercise 10

We will first show that subspaces contain derivatives and then that all coefficients of monomials are contained.

Let V be a finite-dimensional complex vector space and W a sub-vector space. Let $F : (-\varepsilon, \varepsilon) \to W$ be differentiable at 0. Then

$$F'(0) = \lim_{t\to 0} \frac{F(t) - F(0)}{t} \in W \ ,$$

since each term in the limit lies in W by virtue of W being a linear subspace, and the limit lies in W as finite-dimensional sub-vector spaces are closed.
For $F(\lambda_1, \dots, \lambda_n)$ one can extract the coefficients by iterated differentiation. For example, the coefficient of $\lambda_1 \cdots \lambda_n$ is given by

$$\frac{d}{d\lambda_1} \cdots \frac{d}{d\lambda_n} F(\lambda_1, \dots, \lambda_n)|_{\lambda_1=\cdots=\lambda_n=0} \ ,$$

and is hence contained in $\mathrm{span}_{\mathbb{C}}\{H^{\otimes n} | H \in GL(V)\}$.

Exercise 11

The QR-decomposition for $A \in \mathrm{Mat}(N, \mathbb{C})$ states that we can write $A = QR$ with $Q \in U(n)$ and R upper triangular.
If A is invertible, then so is R, i.e. all its diagonal entries are non-zero. If $\det A = 1$ we may also take $\det R = 1$. Pick a path γ such that $\gamma(0) = R$ and $\gamma(1) = D$ where D is the diagonal part of R. This can be done by linearly shrinking all off-diagonal entries to zero. Since we are not changing the diagonal, the path stays in $GL(N, \mathbb{C})$ or $SL(N, \mathbb{C})$, respectively. (What fails in this argument if we start from a matrix that is not upper triangular, e.g. if we were to use A directly instead of R?) Thus A is connected to QD.
We can now pick a path γ in $(\mathbb{C}^\times)^n \hookrightarrow \mathrm{Mat}(N, \mathbb{C})$ such that $\gamma(0) = D$ and $\gamma(1) = Id$. Again, if $\det D = 1$ we can choose γ to stay inside diagonal matrices of determinant 1. Thus A is connected to Q.
If $\det(Q) = e^{i\varphi}$, the path $\gamma(t) = e^{-i\varphi t/N} Q$ is in $U(N)$ and satisfies $\gamma(0) = Q$ and $\det(\gamma(1)) = \det(e^{-i\varphi/N} Q) = 1$, so that $\gamma(1) \in SU(N)$. Thus A is connected to an element of $SU(N)$.

It remains to show that $SU(N)$ is connected. We will do this in two ways, each of which will show that every element in $SU(N)$ can be connected to the identity by a path in $SU(N)$.

Via induction on N:
For $N = 2$ we have $SU(2) = S^3$, the three-sphere, as manifolds. Hence $SU(2)$ is connected.

Suppose we have already shown that $SU(N-1)$ is connected for $N \geq 3$. Let $A \in SU(N)$ be given. Write a_N for the N'th column vector of A and e_N for the N'th basis vector of the standard basis of $\mathbb{C}^N$.
If $a_N = \lambda e_N$ for some $\lambda \in \mathbb{C}^\times$, we are done as then A is block-diagonal with a block of size $N-1$, call it A', and a block of size 1 containing λ. As before we can pick a path in $SU(N)$ which sets λ to 1 by multiplying with appropriate diagonal matrices. Thus A is connected to a block matrix with a block of size

$N-1$ containing an element of $SU(N-1)$ and a block of size 1 containing a 1, which is connected to Id by induction hypothesis.
If a_N is not proportional to e_N, consider the vectors

$$v_1 = e_N \quad , \quad v_2 = (a_N - A_{NN}e_N)/|a_N - A_{NN}e_N| \ .$$

These are orthonormal by construction. Complete them to an orthonormal basis $v_1, \dots, v_N$ and let $U \in U(N)$ denote the corresponding change of basis, i.e. $Ue_1 = v_1$, etc. In terms of the new basis, we have

$$a_N = xv_1 + yv_2 \ ,$$

where $x = A_{NN}$ and $y = |a_N - A_{NN}e_N|$. Note that $x^2 + y^2 = 1$ (as it must be as a_N has unit length). Pick an element $Z \in SU(2)$ which maps (x, y) to $(1, 0)$. This is connected to the identity (as $SU(2)$ is connected by the base case). Let γ be a path with $\gamma(0) = Id$, $\gamma(1) = Z$.
By abuse of notation, we write γ for the path in $SU(N)$ where the 2×2 matrices are placed in the upper right corner.
Set $\tilde{\gamma}(t) = U\gamma(t)U^{-1}$. Then $\tilde{\gamma}(0) = Id$ and

$$\tilde{\gamma}(1)a_N = UZU^{-1}(xv_1 + yv_2) = UZ(xe_1 + ye_2) = Ue_1 = v_1 = e_N \ .$$

Hence

$$t \mapsto \tilde{\gamma}(t)A$$

is a path in $SU(N)$ which is A for $t = 0$ and which satisfies, for $A' := \tilde{\gamma}(1)A$, that $A'e_N = \tilde{\gamma}(1)a_N = e_N$. Hence A' is now block-diagonal.

Via diagonalisation: (much shorter, but I did not think of it before seeing it in the exercise class)
Let $A \in SU(N)$ be given. We can find $U \in U(N)$ such that $A = UDU^{-1}$ where D is a diagonal matrix. Since all diagonal entries of D have modulus 1, we can write $D = \mathrm{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_N})$ for some $\lambda_j \in \mathbb{R}$. Since $\det(D) = \det(A) = 1$ we may choose the λ_j such that $\lambda_1 + \dots + \lambda_N = 0$. The path

$$\gamma(t) = UD(t)U^{-1} \quad , \quad D(t) = \mathrm{diag}(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t})$$

is in $SU(N)$ for all t and satisfies $\gamma(1) = A$, $\gamma(0) = Id$.

Exercise 12

1. Tensoring together elements of the standard basis of $\mathbb{C}^N$ gives a basis on which D acts diagonal. If D has entries $D_1, \dots, D_N$ along the diagonal, then

   $$D.(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}) \ = \ D_{i_1}D_{i_2}\cdots D_{i_N} \cdot e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N} \ .$$

   Let λ be a Young diagram with n boxes and write $\pi = (-).\hat{c}_\lambda$ for the projection $\pi : (\mathbb{C}^N)^{\otimes n} \to S_\lambda(\mathbb{C}^N)$. Since π is a $GL(N, \mathbb{C})$ intertwiner, the $\pi(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N})$ are eigenvectors for the D action on $S_\lambda(\mathbb{C}^N)$. They clearly span $S_\lambda(\mathbb{C}^N)$, and so by omitting vectors one can find a basis of $S_\lambda(\mathbb{C}^N)$ consisting of eigenvectors of the D-action.

2. We saw in the lecture that d acts by $\sum_{j=1}^{n} d_j$. Let $\delta_1, \ldots, \delta_N$ be the diagonal entries of d. Then

$$d.(e_{i_1} \otimes \cdots \otimes e_{i_N}) \ = \ (\delta_{i_1} + \cdots + \delta_{i_N}) \cdot e_{i_1} \otimes \cdots \otimes e_{i_N} \ .$$

In particular E_{11} countes the number of times e_1 occurs. Hence the maximal eigenvalue is attained by $w := e_1 \otimes e_1 \otimes \cdots \otimes e_1$. The eigenvalue is n and the corresponding eigenspace is one-dimensional and spanned by w.

Let π be as in part 1. Then $c_\lambda = a_\lambda b_\lambda$ and a_λ just multiplies w by the order of P_λ. But $w.b_\lambda$ is zero unless $b_\lambda = 1$, which happens if and only if $Q_\lambda = \{id\}$, i.e. if there is exactly one row.

Thus the vector w is contained only in $\mathrm{Sym}^n(\mathbb{C}^N)$.