Solutions for exercise sheet # 02
Topics in representation theory WS 2017

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Exercise 5

1. (a) By definition, $b = \frac{1}{n!}\sum_{g\in S_n} \mathrm{sgn}(g) e_{g(1)} \otimes \cdots \otimes e_{g(n)}$. All the vectors in this sum are linearly independent, and so in particular, $b \neq 0$.

   Write $u = e_{i_1} \otimes \cdots \otimes e_{i_n}$. If two indicies in $(i_1, \ldots, i_n)$ are equal, $\pi(u) = 0$ as we have $t.u = u$ for a transposition exchanging these two tensor factors. But $\pi(t.u) = -\pi(u)$, and so $\pi(u) = 0$. If all indices are distinct, there is a unique $h \in S_n$ such that $u = h.(e_1 \otimes \cdots \otimes e_n)$. We get

   $$\pi(u) = \mathrm{sgn}(h)\pi(e_1 \otimes \cdots \otimes e_n) = \mathrm{sgn}(h)\, b\ .$$

   Thus, for every basis vector u of $V^{\otimes n}$, $\pi(u) \in \mathbb{C}b$. Hence $\mathrm{im}(\pi) = \mathbb{C}b$ is one-dimensional.

   (b) Let $F(e_i) = \sum_j F_{ji} e_j$ be the matrix representation of $F \in GL(V)$. Then

   $$\begin{aligned} F.b &\overset{(1)}{=} F.\pi(e_1 \otimes \cdots \otimes e_n) \\ &\overset{(2)}{=} \pi(F.(e_1 \otimes \cdots \otimes e_n)) = \pi(F(e_1) \otimes \cdots \otimes F(e_n)) \\ &\overset{(3)}{=} \sum_{i_1,\ldots,i_n} F_{1i_1} \cdots F_{ni_n} \pi(e_{i_1} \otimes \cdots \otimes e_{i_n}) \\ &\overset{(4)}{=} \sum_{h\in S_n} F_{1h(1)} \cdots F_{nh(n)} \pi(e_{h(1)} \otimes \cdots \otimes e_{h(n)}) \\ &\overset{(5)}{=} \sum_{h\in S_n} F_{1h(1)} \cdots F_{nh(n)} \mathrm{sgn}(h) b = \det(F) b\ . \end{aligned}$$

   Here, (1) is the definition of b; (2) uses that π is a $GL(V)$-intertwiner; in (3) the expression of F in a basis is substituted; in (4) we use part (a) which say that either π gives zero, or there is a unique permutation h which gives the indices; (5) is again a result of part (a).

2. Let us choose a Young tableau T for λ where the boxes in each column are labelled in increasing order.

   Claim: The image of $(-).b_T$ is $\mathbb{C}\, b \otimes \cdots \otimes b$ (k factors).

   Write π for the antisymmetriser on n factors as in part 1a. Let $v = v_1 \otimes \cdots \otimes v_k$, where each $v_i \in V^{\otimes n}$. Then one checks that $v.b_T = (const)\, b \otimes \cdots \otimes b$.

   Since the image of $(-).b_T$ is one-dimensional, that of $(-).c_T$ is either zero or one-dimensional. But the vector $b \otimes \cdots \otimes b$ is invariant under a_T (up to a constant), and so the image of $(-).c_T$ is one-dimensional.

Finally $F.b^{\otimes k} = (F.b)^{\otimes k} = \det(F)^k\, b^{\otimes k}$.

Exercise 6

1. Just write out the conditions, using that $e_{g^{-1}} e_{h^{-1}} = e_{(hg)^{-1}}$ and $\mathrm{sgn}(g) e_g \mathrm{sgn}(h) e_h = \mathrm{sgn}(gh) e_{gh}$.

2. We define T^t to be a reflection along the top-right to bottom-left diagonal. In particular, rows and columns are exchanged:

$$P_T = Q_{T^t} \quad , \quad Q_{T^t} = P_T \ .$$

   Then e.g.

$$\begin{aligned}\varphi(b_T) &= \varphi(\sum_{g \in Q_T} \mathrm{sgn}(g) e_g) = \sum_{g \in Q_T} \mathrm{sgn}(g) \varphi(e_g) \\ &= \sum_{g \in Q_T} \mathrm{sgn}(g)^2 \varphi(e_{g^{-1}}) = \sum_{h \in P_{T^t}} \varphi(e_h) = a_{T^t} \ .\end{aligned}$$

3. Let $c_T x \in \tilde{V}_T$. Then

$$\varphi(c_T x) = \varphi(a_T b_T x) = \varphi(x)\varphi(b_T)\varphi(a_T) = \varphi(x) a_{T^t} b_{T^t} = \varphi(x) c_{T^t} \in V_{T^t} \ .$$

   Thus $\varphi(\tilde{V}_T) \subset V_{T^t}$. For the other inclusion one can repeat the above computation to get $\varphi^{-1}(V_{T^t}) \subset \tilde{V}_T$.

   The above computation also shows that the linear isomorphism $\varphi : \tilde{V}_T \to V_{T^t}$ satisfies $\varphi(ux) = \varphi(x)\varphi(u)$, i.e. after precomposing the left Γ-actin on V_{T^t} with φ and thinking of it as a right module, it is isomorphic to $\tilde{V}_T$.

4. Let us do the second part of Theorem 1.1.4 as an example, the other parts are similar. Let $S \subset \tilde{V}_T$ be an invariant subspace. Then $\varphi(S)$ is an invariant subspace of V_{T^t}: Let $s \in \varphi(S)$ and $x \in \Gamma$. Then

$$x\varphi(s) = \varphi(\varphi^{-1}(x))\varphi(s) = \varphi(s\varphi^{-1}(x)) \in \varphi(S)$$

   where in the last step we used that S is invariant with respect to the right action by Γ. But V_{T^t} is irreducible by Theorem 1.1.4. Hence $\varphi(S) = \{0\}$ or $\varphi(S) = V_{T^t}$, giving $S = \{0\}$ or $S = \tilde{V}_T$.

Exercise 7

1. Let $a \in A$. We need to show that $a \in A''$. Let $y \in A'$ be arbitrary. Then $ya = ay$ by definition of A' and so $a \in A''$.

2. Clear.

3. We have $A' \subset A'''$ by part 1. By part 2, $A \subset A''$ implies $A' \supset A'''$. Hence they are equal.

4. Clearly $1 \in A'$ and A' is a sub-vector space (its defining condition is linear). Finally, if $a, b \in A'$, then for all $x \in A$: $xab = axb = abx$, and so $ab \in A'$.

Exercise 8

Write $\Gamma_k = \mathbb{C}S_k$. The proof is similar to that of Lemma 1.2.6. Write

$$\psi : S_\lambda V \otimes S_\mu V \to \mathrm{Hom}_{\Gamma_{m+m}}(\tilde{V}_\lambda \bullet \tilde{V}_\mu, V^{\otimes(m+n)}) \quad , \quad \psi(u) = (x \mapsto u.x) \ .$$

$\psi(u)$ is a Γ_{m+n}-intertwiner by the same argument as in Lemma 1.2.6.
For surjectivity let $\varphi : \tilde{V}_\lambda \bullet \tilde{V}_\mu \to V^{\otimes(m+n)}$ be given. By definition $\hat{c}_\lambda \otimes \hat{c}_\mu \in \tilde{V}_\lambda \bullet \tilde{V}_\mu$ and so we can set $u := \varphi(\hat{c}_\lambda \otimes \hat{c}_\mu)$. The rest of the computation is the same as in Lemma 1.2.6.
For injectivity, suppose $\psi(u) = 0$. Then in particular $0 = \psi(u)(\hat{c}_\lambda \otimes \hat{c}_\mu) = u.(\hat{c}_\lambda \otimes \hat{c}_\mu)$. But u itself can be written as $v.(\hat{c}_\lambda \otimes \hat{c}_\mu)$ for some $v \in V^{\otimes m} \otimes V^{\otimes n}$. Combining this, we conclude $u = 0$.

Regarding the aside: We have

$$\tilde{V}_\lambda \bullet \tilde{V}_\mu \cong (\tilde{V}_\lambda \otimes_{\mathbb{C}} \tilde{V}_\mu) \otimes_{\Gamma_m \otimes \Gamma_n} \Gamma_{m+n} \ ,$$

i.e. $\tilde{V}_\lambda \bullet \tilde{V}_\mu$ is the induced Γ_{m+n}-right module $\mathrm{Ind}_{\Gamma_m \otimes \Gamma_n}^{\Gamma_{m+m}}(\tilde{V}_\lambda \bullet \tilde{V}_\mu)$.
To see this we write module homomorphisms in both ways which are inverse to each other. Let L be the lhs and R be the rhs above, and define $f : L \to R$ and $g : R \to L$ as

$$f((\hat{c}_\lambda \otimes \hat{c}_\mu)x) = (\hat{c}_\lambda \otimes \hat{c}_\mu) \otimes_{\Gamma_m \otimes \Gamma_n} x$$

and

$$g((\hat{c}_\lambda a \otimes \hat{c}_\mu b) \otimes_{\Gamma_m \otimes \Gamma_n} x) = (\hat{c}_\lambda \otimes \hat{c}_\mu)abx \ .$$

One now needs to check that these maps are indeed well-defined, land in the correct spaces and are intertwiners of Γ_{m+n}-right modules. It is then clear that they are inverse to each other.