**Solutions for exercise sheet # 01**
**Topics in representation theory WS 2017**

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**Exercise 1**

1. There cannot be a non-trivial $GL(V)$-invariant subspace: If $S \neq \{0\}$ is a proper subspace of $V$, then there is a nonzero $x \in S$ and a non-zero $y \in V \setminus S$. But then one can find an invertible linear map that takes $x$ to $y$ (extend $x$ to a basis and $y$ to a basis and use the transformation between these), i.e. $S$ is not invariant.

2. Invariance is clear.

   To show the direct sum decomposition, set $\pi_\pm : V \otimes V \to V \otimes V$, $\pi_\pm(x \otimes y) = \frac{1}{2}(x \otimes y \pm y \otimes x)$. Then the image of $\pi_+$ is $\mathrm{Sym}^2(V)$ and that of $\pi_-$ is $\mathrm{Alt}^2(V)$.

   Note that $\pi_\pm$ are orthogonal idempotents: $\pi_+ + \pi_- = id$, $\pi_+\pi_+ = \pi_+$, $\pi_-\pi_- = \pi_-$, $\pi_+\pi_- = 0 = \pi_-\pi_+$.

   Any $u \in V \otimes V$ can be decomposed as $u = \pi_+(u) + \pi_-(u)$, showing $V \otimes V = \mathrm{Sym}^2(V) + \mathrm{Alt}^2(V)$. Furthermore, if $u \in \mathrm{Sym}^2(V) \cap \mathrm{Alt}^2(V)$, then $u = \pi_+(u)$ and $u = \pi_-(u)$, and so $u = \pi_+\pi_-(u) = 0$. Hence the sum is direct.

**Exercise 2**

There are several ways to do this. For the first way, fix a field $k$ and a $k$-vector space $V$. Write

$$X_V = \{\rho : G \to GL(V) | \rho \text{ is a group homomorphism } \}$$

and

$$Y_V = \{\sigma : kG \to \mathrm{End}(V) | \sigma \text{ is an algebra homomorphism } \} .$$

*Claim*: The map $f_V : Y_V \to X_V$ with $f_V(\sigma)(g) = \sigma(e_g)$ is well-defined and a bijection.

*Proof.* For well-definedness we need to check that $f_V(\sigma)$ is indeed an element in $X_V$. Write $\rho = f_V(\sigma)$. That $\rho(g)$ is in $GL(V)$ follows from $e_g e_{g^{-1}} = 1$, which shows that $\sigma(e_g)$ is invertible. That $\rho(1) = id$ and $\rho(gh) = \rho(g)\rho(h)$ is immediate from the corresponding properties of $\sigma$.
To see that $f_V$ is a bijection, consider the map $g_V : X_V \to Y_V$, $g_V(\rho)(\sum_{h \in G} \lambda_h e_h) := \sum_{h \in G} \lambda_h e_{\rho(h)}$. One checks $g_V(\rho)$ is indeed in $Y_V$ and that this map is inverse to $f_V$. $\square$
For intertwiners one has the following statement. Let $(V, \rho_V)$ and $(W, \rho_W)$ be $G$-representations and $\phi : V \to W$ a linear map.

*Claim*: $\phi$ is a $G$-intertwiner if and only if it is an intertwiner of the $kG$ representations $(V, f_V(\rho_V))$ and $(W, f_W(\rho_W))$.

The proof consists of writing out the second condition in the standard basis $\{e_h\}$ of $kG$ and comparing it to the first.

Another way to make a precise statement of the exercise is to define a functor from $G$-representations to $kG$-modules via $(V, \rho) \mapsto (V, f_V(\rho))$ and as the identity on morphisms, to show that this is indeed a functor between the claimed categories, and to show that it is an equivalence (an isomorphism, in fact, i.e. one can write an inverse functor without the need of additional natural isomorphisms to the identity functor).

**Exercise 3**

1. That irreducible implies indecomposable is clear. The converse is in general false: Take the group $(\mathbb{R}, +)$ and the two-dimensional complex representation $V$ given by $r \mapsto \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$. $\mathbb{C}(1,0)$ is an invariant subspace, so $V$ is reducible. However, if there were a direct sum decomposition into two invariant subspaces, the matrix $\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ would be diagonalisable for all $r$, which it is not.

2. Let $\Gamma = \mathbb{C}G$ be the group algebra. Since $G$ is finite, $\Gamma$ is finite dimensional. Let $V$ be an infinite dimensional representation of $G$ and hence of $\Gamma$. Let $v \in V$ be a non-zero vector. Then $W := \Gamma.v$ is an invariant subspace. Since $\dim(W) \le \dim(\Gamma)$, this subspace is finite dimensional, and hence not all of $V$. Since $v \in W$, we also have that $W \neq \{0\}$. Hence $V$ is reducible.

   Pick a projection $\pi : V \to W$ (not necessarily a $G$-intertwiner). Define $\bar{\pi} : V \to W$ as, for $x \in V$,

   $$\bar{\pi}(x) = \frac{1}{|G|} \sum_{g \in G} g.\pi(g^{-1}.x) \ .$$

   Then one checks that $\bar{\pi}$ is a $G$-intertwiner with $\bar{\pi}\bar{\pi} = \bar{\pi}$ and $\bar{\pi}|_W = id_W$, i.e. it is a projection to $W$. Projections satisfy $V = \mathrm{im}(\bar{\pi}) \oplus \ker(\bar{\pi})$, and since $\bar{\pi}$ is an intertwiner, $\ker(\bar{\pi})$ is a subrepresentation.

   Hence $V$ is also decomposable.

**Exercise 4**

1. The canonical Young tableau has $1, 2$ in the first row and $3$ in the second. Hence $a_T = 1 + e_{(12)}$ and $b_T = 1 - e_{(13)}$. Thus

   $$c_\lambda = (1 + e_{(12)})(1 - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(321)}$$

2. $V_\lambda$ is spanned $e_g c_\lambda$ where $g$ runs over the 6 basis elements of $S_3$. Since $e_{(12)} a_T = a_T$ it is enough to consider the three elements $1, (13), (23)$.

$$\begin{aligned} 1 \cdot c_\lambda &= 1 + e_{(12)} - e_{(13)} - e_{(321)} \\ e_{(13)} \cdot c_\lambda &= e_{(13)} + e_{(123)} - 1 - e_{(23)} \\ e_{(23)} \cdot c_\lambda &= e_{(23)} + e_{(321)} - e_{(123)} - e_{(12)} \end{aligned}$$

But $c_\lambda + e_{(13)} c_\lambda = -e_{(23)} c_\lambda$, and so these vectors span a 2-dimensional space.