Exercise sheet # 12
Topics in representation theory WS 2017

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Exercise 50

1. Check that a super algebra with super commutator is a Lie super algebra.
2. Show that there is an even isomorphism $gl(V) \to gl(\Pi V)$ of Lie super algebras

Exercise 51

Verify the graded Jacobi identity in the reconstruction of a Lie super algebra from the data listed in Prop. 4.1.7.

Exercise 52

Let V be a super-vector space and $f : V \to V$ a linear map. Let $\pi_\varepsilon : V \to V_\varepsilon$ be the projection onto the two graded components of V. Define the *super-trace* of f to be

$$\mathrm{str}_V(f) = \mathrm{tr}_{V_0}(\pi_0 f|_{V_0}) - \mathrm{tr}_{V_1}(\pi_1 f|_{V_1}) \ .$$

Define

$$sl(m|n) = \big\{ A \in gl(m|n) \,\big|\, \mathrm{str}(A) = 0 \big\} \ .$$

Is this a Lie-super algebra (via the super commutator)? What if one uses the ordinary trace instead of the super-trace?

Exercise 53

Show that $\varphi : so(r,s) \to Cl^0_{r,s}$, $A \mapsto -\frac{1}{4}\sum_{i,j=1}^{r+s}(A\eta)_{ij} e_i e_j$ is a Lie algebra homomorphism.

Exercise 54

Let k be a field of characteristic zero and (V,q) a quadratic vector space. Show that the map

$$s : \Lambda(V) \to Cl(V,q) \quad , \quad v_1 \wedge \cdots \wedge v_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(m)}$$

is an isomorphism of k-vector spaces.