Exercise sheet # 11
Topics in representation theory WS 2017

(Ingo Runkel)

Exercise 45

Show that $Pin_{1,1}$ is a trivial double cover of $O_{1,1}$.

Exercise 46

Let U, U' be finite-dimensional $\mathbb{C}$-vector spaces and and set $S = \Lambda(U')$. Let $\beta : U \times U' \to \mathbb{C}$ be bilinear and let $x \in U$. Show that there exists a unique $\mathbb{C}$-linear map $x \lrcorner (-) : S \to S$ such that:

1. for all $y \in U'$ we have $x \lrcorner y = \beta(x, y)$,
2. for all $a \in \Lambda^r(U')$, $b \in \Lambda^s$ we have $x \lrcorner (a \wedge b) = (x \lrcorner a) \wedge b + (-1)^r a \wedge (x \lrcorner b)$.

Exercise 47

Show Lemma 3.5.9: There is a unique algebra homomorphism ρ such that

$$\begin{tikzcd} U \oplus U' \arrow[r] \arrow[d, "(\lrcorner, \wedge)"'] & Cl(U \oplus U', q) \arrow[dl, "\rho"] \\ \mathrm{End}(\Lambda(U')) & \end{tikzcd}$$

commutes.

Exercise 48

Show Lemma 3.6.4: Let A be an $\mathbb{R}$-algebra and let (W, ρ) be a finite-dimensional representation of A over $\mathbb{C}$. The following are equivalent:

1. W is real.
2. There is a $\mathbb{C}$-basis $w_1, \ldots, w_n$ of W such that the matrix elements of $\rho(a)$ with respect to this basis are real for all $a \in A$.

Please turn over.

Exercise 49

Show a part of Rem 3.6.3: Let A be a semisimple $\mathbb{R}$-algebra and let (V, ρ) be a finite-dimensional irreducible representation of A over $\mathbb{R}$. Pick your favourite of the following three statements and prove it:

1. If $\mathrm{End}_{A,\mathbb{R}}(V) \cong \mathbb{R}$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is an irreducible A-representation over $\mathbb{C}$.
2. If $\mathrm{End}_{A,\mathbb{R}}(V) \cong \mathbb{C}$, then V can be made into a $\mathbb{C}$-vector space such that V becomes a A-representation over $\mathbb{C}$ in exactly two non-isomorphic ways.
3. If $\mathrm{End}_{A,\mathbb{R}}(V) \cong \mathbb{H}$, then V can be made into a $\mathbb{C}$-vector space such that V becomes a A-representation over $\mathbb{C}$, and any two ways of doing this lead to representations that are isomorphic (as representations over $\mathbb{C}$).

Hint:
For parts 2 and 3, what do you know about the automorphisms of $\mathbb{C}$ and $\mathbb{H}$?