Exercise sheet # 10
Topics in representation theory WS 2017

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Exercise 39

1. Show Prop. 3.2.3: Let

   $$Cl(V,q) = T(V)/\langle\, v\otimes v + q(v)1 \,|\, v \in V\,\rangle \quad , \quad \iota = \big[V \xrightarrow{\iota_T} T(V) \xrightarrow{\pi} Cl(V,q)\big] \ .$$

   Show that $Cl(V,q)$ together with ι satisfies the universal property of Clifford algebras.

2. Let (V,p) and (W,q) be K-vector spaces with quadratic forms. Let $f : V \to W$ be K-linear s.th. $f^*q = p$. In the remark after Prop. 3.2.3 we used the universal property to define an algebra homomorphism $Cl(f) : Cl(V,p) \to Cl(W,q)$ via the diagram:

   $$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[d, "f"'] & Cl(V,q) \arrow[ddl, dashed, "\exists! Cl(f)"] \\ W \arrow[d, "\iota"'] & \\ Cl(W,q) & \end{tikzcd}$$

   Show that $Cl(f_1) \circ Cl(f_2) = Cl(f_1 \circ f_2)$ for appropriately defined f_1, f_2.

Exercise 40

1. What is the centre of $\mathbb{H}$?

2. Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ does not have any non-trivial two-sided ideals.

   Hint: Show that a non-zero ideal I contains $p \otimes 1$ for some $p \neq 0$. This can be done as follows: Let $x \in I$ be non-zero and write $x = \sum_a p_a \otimes q_a$. Assume that x is such that this decomposition uses the minimal number of summands amongst the non-zero elements in I. Why are the p_a linearly independent (over $\mathbb{R}$)? Let $x' = x \cdot (1 \otimes q_1^{-1})$, so that $x' = p_1 \otimes 1 + \dots$. Now write out $(1 \otimes y)x' - x'(1 \otimes y)$ and stare at the result.

   Voluntary aside: The above proof strategy does not really use any specifics of the quaternions. Can you formulate a more general statement?

Please turn over.

Exercise 41

1. Show that $Cl_{2,0} \cong \mathbb{H}$ and that $Cl_{0,2} = \mathrm{Mat}(2, \mathbb{R})$.
2. Show that the map

$$\psi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H}) \quad , \quad p \otimes q \longmapsto (x \mapsto px\overline{q})$$

   is an isomorphism of $\mathbb{R}$-algebras. Conclude that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}(4, \mathbb{R})$ as $\mathbb{R}$-algebras.

   Note: To show bijectivity, there is the elementary but tedious method to show that ψ maps a basis (consisting of 16 elements) to a basis. Can you think of a less tedious method?

Exercise 42

Prove Theorem 3.3.6 on the classification of $Cl_{r,s}$ in terms of matrix algebras.

Exercise 43

Show that all $\mathbb{R}$-algebra automorphisms of $\mathbb{H}$ are inner, that is, they are of the form $q(-)q^{-1}$ for some $q \in \mathbb{H} \setminus \{0\}$.

For example, you could proceed along the following lines:

1. $\mathbb{H}$ is a simple $\mathbb{H}$-$\mathbb{H}$-bimodule. Why is this the same as a simple $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$-left module in this case? Why is there only one such simple left module up to isomorphism? (Recall theorem 1.2.9.)
2. Let α be an $\mathbb{R}$-algebra automorphism of $\mathbb{H}$. Denote by ${}_{\alpha}\mathbb{H}$ the bimodule where the left copy of $\mathbb{H}$ acts by first composing with α (while the right copy still acts by multiplication from the right). By part 1 there exists a bimodule isomorphism $\mathbb{H} \to {}_{\alpha}\mathbb{H}$. Ponder this fact.