Exercise sheet # 09
Topics in representation theory WS 2017

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Exercise 35

Let V be a K-vector space and let $T(V) = \bigoplus_{m=0}^{\infty} T^m$ with $T^m = V^{\otimes m}$ have the algebra structure as defined in the lecture. Let $i : V \to T(V)$ be the embedding into the summand T^1.

Show that $(T(V), i)$ satisfies the universal property of a tensor algebra.

Exercise 36

Let $(T(V), i)$ be the tensor algebra as in Exercise 35.

1. Show that there is a unique algebra homomorphism $\Delta : T(V) \to T(V) \otimes T(V)$ which satisfies $\Delta(i(x)) = i(x) \otimes 1 + 1 \otimes i(x) \in (T^1 \otimes T^0) \oplus (T^0 \otimes T^1)$.

2. Compute $\Delta(i(x)i(y))$ for $x, y \in V$.

3. Show that $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$.

Exercise 37

Let A be a K-algebra and let $I \subset A$ be a two-sided ideal.

1. Consider the quotient K-vector space A/I and the canonical projection $\pi : A \to A/I$, $a \mapsto a + I$. Show that there is a unique (associative unital) algebra structure on A/I such that π is an algebra homomorphism.

2. Look at the diagram below and formulate (and prove) the universal property of A/I:

$$\begin{tikzcd}
I \arrow[r, hook] \arrow[dr, "0"'] & A \arrow[r, "\pi"] \arrow[d, "f"] & A/I \arrow[dl, dashed, "\tilde{f}"] \\
& B &
\end{tikzcd}$$

Please turn over.

Exercise 38

Recall from the lecture the universal property of the alternating algebra $\Lambda(V)$ of a vector space V over some field K.

1. Show that an alternating algebra is unique up to unique isomorphism.
2. Show that $\Lambda(V) := T(V)/\langle v \otimes v \mid v \in V\rangle$ with $i_\Lambda = \left[V \xrightarrow{i_T} T(V) \xrightarrow{\pi} \Lambda(V)\right]$ is an alternating algebra.
3. Let $\Lambda^m := \pi(T^m)$. Show that $\Lambda(V) = \bigoplus_{m=0}^{\infty} \Lambda^m$.
4. Let $K = \mathbb{C}$ and let $\lambda = (1, 1, \ldots, 1)$ be the Young-diagram consisting of a single column with m boxes. Show that $S_\lambda(V) \cong \Lambda^m$ as k-vector spaces. (Note that $S_\lambda(V)$ is defined as a sub-vector space of $V^{\otimes m}$, while Λ^m is defined as a quotient of $V^{\otimes m}$.) Are they isomorphic as $GL(V)$-modules?