Exercise sheet # 08
Topics in representation theory WS 2017

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Exercise 30

Prove Lemma 2.6.5: Recall the orbit $\mathcal{O}_m^+$ from the lecture, the diffeomorphism $\pi : \mathcal{O}_m^+ \to \mathbb{R}^3$, and the definition of $\widetilde{\Lambda}(\vec{p}) = \pi(\Lambda(\pi^{-1}(\vec{p})))$, where $\vec{p} \in \mathbb{R}^3$ and $\Lambda \in L_+^\uparrow$. For $\lambda > 0$, we defined

$$\mu(\vec{p}) = \frac{\lambda}{p_0} = \frac{\lambda}{\sqrt{m^2 + |\vec{p}|^2}} \ .$$

We need to show that μ transforms under the action of $L_+^\uparrow$ as a density, that is, for all $\Lambda \in L_+^\uparrow$,

$$\mu(\widetilde{\Lambda}(\vec{p})) = \frac{1}{|\det D\widetilde{\Lambda}(\vec{p})|}\, \mu(\vec{p}) \qquad (*)$$

1. Suppose you did already prove $(*)$. Let $f : \mathbb{R}^3 \to \mathbb{C}$ be continuous and have compact support. Abbreviate $I(f) = \int_{\mathbb{R}^3} f(x)\mu(x) d^3x$. Show that for all $\Lambda \in L_+^\uparrow$,
$$I(f \circ \widetilde{\Lambda}) = I(f) \ .$$
2. Show that $(*)$ holds for rotations $R \in SO(3) \subset L_+^\uparrow$, for boosts, and finally for all $\Lambda \in L_+^\uparrow$.
3. Why is a function μ satsifying (*) unique up to a constant?

Exercise 31

For $q \in \mathcal{O}_m^+$ we defined the 2×2 matrix T_q as

$$T_q := \frac{1}{\sqrt{2m(m + q_0)}}\, (m \cdot id + \widehat{q}) \ .$$

Show that $T_q \in SL(2, \mathbb{C})$ and that for $a = (m, 0, 0, 0)$ we have $T_q\, \widehat{a}\, T_q^\dagger = \widehat{q}$.

Please turn over.

Exercise 32
For $(x,g) \in \widetilde{P}$ and $f \otimes v \in \mathcal{L} \otimes R$ we defined

$$\big((x,g).(f \otimes v)\big)(\vec{p}) = e^{i\eta(x,\pi^{-1}(\vec{p}))} f(\widetilde{\phi(g^{-1})}\,\vec{p}) \otimes W(\phi(g^{-1})\pi^{-1}(\vec{p}), g).v \ ,$$

where $\vec{p} \in \mathbb{R}^3$

Show that this defines a unitary representation of $\widetilde{P}$. (You do not need to show continuity of the action.)

Hint: The formulas are somewhat simpler if one uses $L^2(\mathbb{R}^3, R)$, that is, L^2-functions on $\mathbb{R}^3$ with values in R, the unitary representation of $SU(2)$ we chose. The inner product on $L^2(\mathbb{R}^3, R)$ is

$$(f,g) = \int_{\mathbb{R}^3} (f(x), g(x))_R \, \mu(x) \, d^3x \ ,$$

where $(-,-)_R$ denotes the inner product on R. By construction we have a unitary equivalence $L^2(\mathbb{R}^3, R) \cong \mathcal{L} \otimes R$.

Exercise 33
Consider the orbit $\mathcal{O}_0^+$ and let $a = (\frac{1}{2}, 0, 0, \frac{1}{2}) \in \mathcal{O}_0^+$.

1. Show that

$$\mathrm{Stab}(a) = \left\{ \begin{pmatrix} e^{i\theta/2} & ze^{-i\theta/2} \\ 0 & e^{-i\theta/2} \end{pmatrix} \middle| \theta \in \mathbb{R}, z \in \mathbb{C} \right\} \quad .$$

2. Recall that $E(2)_0$ denotes the connected component of the identity of the group of euclidean motions in two dimensions. Show that the map $\tau : \mathrm{Stab}(a) \to E(2)_0$,

$$\begin{pmatrix} e^{i\theta/2} & ze^{-i\theta/2} \\ 0 & e^{-i\theta/2} \end{pmatrix} \quad \mapsto \quad (z, e^{i\theta}) \ ,$$

is a surjective group homomorphism. What is its kernel?

Exercise 34
Compute the little group for all 6 types of orbits of $L_+^\uparrow$ on 4-dimensional Minkowski space. For which orbits does the little group possess finite-dimensional unitary representations?