# Exercise sheet # 07
# Topics in representation theory WS 2017

(Ingo Runkel)

---

**Exercise 26**

1. Write the Poincaré group $P$ as the semidirect product $\mathbb{R}^4 \rtimes L$.
2. Recall the map $\phi : SL(2,\mathbb{C}) \to L_+^\uparrow$ defined in Section 2.5. Show that $\phi$
   (a) indeed has image in $L_+^\uparrow$ (you do not need to show that it is surjective),
   (b) is a group homomorphism,
   (c) has kernel $\{\pm id\}$.

**Exercise 27**
Let $\mathcal{H}$ be a Hilbert space of dimension $\geq 2$ and let $G$ be a connected Lie group satisfying (*) in Theorem 2.5.5. In Proposition 2.4.2 and Corollary 2.5.6 we had the commuting diagrams

$$
\begin{tikzcd}
U(1) \arrow[r] \arrow[d, equal] & \widehat{G} \arrow[r, "p"] \arrow[d, dashed, "\widehat{\rho}"] & G \arrow[d, "\rho"] \\
U(1) \arrow[r] & U(\mathcal{H}) \arrow[r, "\gamma"] & U(\mathbb{P}(\mathcal{H}))
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\pi_1(G) \arrow[r] \arrow[d, dashed, "\zeta"] & \widetilde{G} \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\rho}"] & G \arrow[d, "\rho"] \\
U(1) \arrow[r] & U(\mathcal{H}) \arrow[r, "\gamma"] & U(\mathbb{P}(\mathcal{H}))
\end{tikzcd}
$$

Is there a commuting diagram you can build from the two upper exact rows? If so, what can you say about the injectivity/surjectivity properties of the group homomorphisms you found?

**Exercise 28**

1. Show the implications
$$\text{(norm cont.)} \overset{(a)}{\Rightarrow} \text{(strongly cont.)} \overset{(b)}{\Rightarrow} \text{(weakly cont.)} \ .$$
2. Consider a Hilbert space $\mathcal{H}$ with ON-basis $\{e_n\}_{n\in\mathbb{N}}$. Let $X \subset \mathbb{R}$ be given by $X = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$. Then $X$ is a metric space with metric inherited from $\mathbb{R}$. Consider the map $f : X \to B(\mathcal{H})$ given by $f(0) = 0$, $f(\frac{1}{n}) = e_{k+n}$. Use this to build a counter-example to the converse implication to $(b)$.

3. Let $\mathcal{H} = L^2(U(1))$. For $\varphi \in \mathbb{R}$ let $T_\varphi : \mathcal{H} \to \mathcal{H}$ be given by $(T_\varphi(f))(\psi) = f(e^{i(\psi+\varphi)})$, i.e. $T_\varphi$ shifts a function around $U(1)$ by some angle $\varphi$. Consider the function
$$T : U(1) \longrightarrow U(\mathcal{H}) \quad , \quad e^{i\varphi} \mapsto T_\varphi \ .$$
Check that $T$ indeed takes values in $U(\mathcal{H})$. Use $T$ to build a counter-example to the converse of implication $(a)$.

*Note:* You may assume $T$ is strongly continuous. (Or you can prove it, if you like, e.g. by using the ON-basis from Exercise 29.)

**Exercise 29**

Let $\mathcal{H} = L^2(U(1))$. For $n \in \mathbb{Z}$ let $e_n \in \mathcal{H}$ be the function $e_n(\varphi) = (2\pi)^{-\frac{1}{2}} e^{in\varphi}$. You may use without proof that $\{e_n\}_{n\in\mathbb{Z}}$ is an ON-basis of $\mathcal{H}$.

1. Define $T_\varphi$ as in Exercise 28 (3). For $m \in \mathbb{Z}$ define $R_m(\sum_{n\in\mathbb{Z}} \lambda_n e_n) := \sum_{n\in\mathbb{Z}} \lambda_n e_{n+m}$. Note that $T_\varphi$ is the "shift in position space" and $R_m$ the "shift in Fourier space".

   Show that $R_m \in U(\mathcal{H})$.

2. Let $G \subset U(\mathcal{H})$ be the subgroup generated by $T_\varphi$, $\varphi \in \mathbb{R}$ and $R_m$, $m \in \mathbb{Z}$. Then $\mathcal{H}$ is a representation of $G$.

   Show that $\mathcal{H}$ is algebraically reducible as a $G$-representation.

   *Hint:* Convince yourself that $R_m$ acts on a function $f$ as $(R_m f)(e^{i\varphi}) = e^{im\varphi} f(e^{i\varphi})$.

3. Show that $\mathcal{H}$ is topologically irreducible as a $G$-representation.

   *Note:* You could use the following result on Hilbert-space valued integrals:[1] Let $I \subset \mathbb{R}$ be a compact interval and let $f : I \to \mathcal{H}$ be a continuous function. Then there exists a unique integral $\int_I f(t)dt \in \mathcal{H}$ with the property that, for all $v \in \mathcal{H}$,
$$\Big(v, \int_I f(t)dt\Big) = \int_I (v, f(t))dt \ .$$
   Here, the second integral is just the Riemann integral of a complex-valued continuous function.

4. Consider now the subgroup of $S \subset G$ generated just by the $R_m$, $m \in \mathbb{Z}$. By part 2, $\mathcal{H}$ is algebraically reducible as an $S$-representation. By Proposition 2.6.2, it must also be topologically reducible (all topologically irreducible $S$-representations being 1-dimensional).

   Can you point at a closed invariant subspace? At a one-dimensional invariant subspace?

[1] See for example Def. 3.26 and Thm. 3.27 in Rudin, Functional Analysis.