Exercise sheet # 06
Topics in representation theory WS 2017

(Ingo Runkel)

Exercise 22

Prove Lemma 2.3.2:

1. A projective representation ρ of G on W gives rise to a group homomorphism $\bar{\rho}$ from G to $PGL(W)$ via $\bar{\rho} = \pi \circ \rho$.

2. Two projective representations (W, ρ) and (W', ρ') are isomorphic (as projective representations) if and only if there is a k-linear isomorphism $f : W \to W'$ such that

$$\begin{tikzcd} & PGL(W) \arrow[d, "\overline{f(\ )f^{-1}}"] \\ G \arrow[ur, "\bar{\rho}"] \arrow[dr, "\bar{\rho}'"'] & \\ & PGL(W') \end{tikzcd}$$

commutes.

3. Let $\sigma : G \to PGL(W)$ be a group homomorphism. Then for any choice of map $\rho : G \to GL(W)$ such that

$$\begin{tikzcd} & GL(W) \arrow[d, "\pi"] \\ G \arrow[ur, dashed, "\rho"] \arrow[r, "\sigma"'] & PGL(W) \end{tikzcd}$$

commutes, (W, ρ) is a projective representation of G. For two different choices ρ and ρ' the projective representations (W, ρ) and (W, ρ') are isomorphic.

Please turn over.

Exercise 23
Show part 2 of Lemma 2.4.1: For all groups H and group homomorphisms a, b such that

$$\begin{tikzcd} H \arrow[r, "a"] \arrow[d, "b"'] & G \arrow[d, "\bar{\rho}"] \\ GL(W) \arrow[r, "\pi"] & PGL(W) \end{tikzcd}$$

commutes, there exists a unique group homomorphism $f : H \to \widehat{G}$ such that

$$\begin{tikzcd} H \arrow[dr, dashed, "f"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\ & \widehat{G} \arrow[r, "p"'] \arrow[d, "\hat{\rho}"] & G \\ & GL(W) & \end{tikzcd}$$

commutes.

Exercise 24

1. Show the following properties of a 2-cocycle χ of G with values in A:
   (a) For all $g \in G$: $\chi(g, e) = \chi(e, e) = \chi(e, g)$ and $\chi(g, g^{-1}) = \chi(g^{-1}, g)$.
   (b) The cohomology class $[\chi]$ contains a normalised 2-cocycle.
2. Complete the proof of Lemma 2.4.4: Show that $A \times_\chi G$ with product as given in the lecture has a unit and an inverse, and that it is a central extension of G by A.

Exercise 25
If G is a subgroup of $GL(W)$ for some vector space W, we will write $PG = G/(G \cap \{\lambda id_W | \lambda \in k^\times\}$. It then makes sense to talk about $PSU(2)$, $PU(2)$, etc.

1. Let $A \xrightarrow{j} G \xrightarrow{p} H$ be a central extension. Suppose there is an element $g \in G$ such that $g \notin j(A)$ but $g^2 \in j(A)$, and such that the (unique) element $a \in A$ with $j(a) = g^2$ cannot itself be written as a square (in A).
   Show that the central extension does not split.
2. The only matrices of the form λid contained in $SU(2)$ are $\pm id$. Show that the central extension $\{\pm 1\} \to SU(2) \to PSU(2)$ does not split (note that here a potential splitting map is not required to be continuous).
3. Show that the central extension $U(1) \to U(2) \to PU(2)$ does not split.
   Hint: $SU(2)$ is generated by commutators $xyx^{-1}y^{-1}$, $x, y \in SU(2)$ (you do not need to prove this).
4. *(This problem is voluntary – no points will be given.)*
   What about $U(1) \to U(N) \to PU(N)$? Does this tell you anything about $U(1) \to UA(\mathcal{H}) \to \mathrm{Aut}\,\mathbb{P}(\mathcal{H})$ for $\mathcal{H} = \mathbb{C}^N$, $N \geq 2$?