Exercise sheet # 05
Topics in representation theory WS 2017

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Exercise 18

Complete the proof of Lemma 2.2.2: Let $\mathcal{H}$ be a Hilbert space, $e, f \in \mathcal{H}$ orthonormal, and $T \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$.

1. Show that there are $e', f' \in \mathcal{H}$ orthonormal such that $T[e] = [e']$ and $T[f] = [f']$.
2. Show that for all $\lambda, \mu \in \mathbb{C}$, not both zero, there are $\lambda', \mu' \in \mathbb{C}$ such that

$$T[\lambda e + \mu f] = [\lambda' e' + \mu' f'] \ .$$

Exercise 19

Complete the proof of Wigner's Theorem at the end of Section 2.2: Let $\mathcal{H}$ and T be as in Exercise 18, and let $e, e' \in \mathcal{H}$ and $V : \mathcal{P} \to \mathcal{P}'$ be defined as in the lecture. Define, for $v = \lambda e + w$ and $w \in \mathcal{P}$,

$$X(v) := \chi(\lambda) e' + V(w) \ .$$

Show that $X \in UA(\mathcal{H})$ and that $T[v] = [Xv]$ holds for all $v \in \mathcal{H} \setminus \{0\}$.

Exercise 20

Let $\mathcal{H}$ be a Hilbert space of dimension ≥ 2. Let G be a group and $\rho : G \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ be a group homomorphism. Recall from Remark 2.1.3 that there is a function $\tilde{\rho} : G \to UA(\mathcal{H})$ such that

$$\begin{tikzcd} & UA(\mathcal{H}) \arrow[d, "\gamma"] \\ G \arrow[ur, dashed, "\tilde{\rho}"] \arrow[r, "\rho"] & \mathrm{Aut}(\mathbb{P}(\mathcal{H})) \end{tikzcd}$$

commutes.

Please turn over.

Show that

1. there is a unique function $\chi : G \times G \to U(1)$ such that $\tilde{\rho}(g)\tilde{\rho}(h) = \chi(g,h)\tilde{\rho}(gh)$ holds for all $g, h \in G$.

2. the function χ satisfies, for all $g, h, k \in G$,

$$\alpha_g(\chi(h,k))\, \chi(gh,k)^{-1}\, \chi(g,hk)\, \chi(g,h)^{-1} = 1 \ ,$$

   where $\alpha_g : \mathbb{C} \to \mathbb{C}$ is given by $\alpha_g(z) = z$ if $\tilde{\rho}(g)$ is unitary and $\alpha_g(z) = \bar{z}$ if $\tilde{\rho}(g)$ anti-unitary.

3. given another function $\tilde{\rho}' : G \to UA(\mathcal{H})$ which makes the above diagram commute, there is a function $\beta : G \to U(1)$ such that for all $g, h \in G$,

$$\chi'(g,h) = \alpha_g(\beta(h))\, \beta(gh)^{-1}\, \beta(g)\, \chi(g,h) \ .$$

   Here, χ' is the function from part 1 corresponding to $\tilde{\rho}'$ and α_g is defined as in part 2.

Exercise 21

Let $\mathcal{H}$ be a Hilbert space. We define the *Fubini-Studi distance function* d on $\mathbb{P}(\mathcal{H})$ as follows:

$$d : \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0, \pi] \quad , \quad \big(\cos \tfrac{d(\alpha,\beta)}{2}\big)^2 = \delta(\alpha, \beta) \ .$$

1. Consider first the case $\mathcal{H} = \mathbb{C}^2$ with its standard inner product. Let e_1, e_2 be the standard basis. Recall from Lemma 2.2.1 the bijection

$$\mathbb{P}(\mathbb{C}^2) \setminus \{\mathbb{C}e_2\} \longrightarrow \mathbb{C} \quad , \quad [e_1 + \lambda e_2] \mapsto \lambda \ .$$

   For the unit 2-sphere $S^2 \subset \mathbb{R}^3$ write $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ and use stereographic projection between $S^2 \setminus (1,0)$ and $\{0\} \times \mathbb{C}$. Show that this gives a bijection

$$\mathbb{C} \longrightarrow S^2 \setminus (1,0) \quad , \quad \lambda \mapsto \frac{1}{|\lambda|^2 + 1}\big(|\lambda|^2 - 1 \,,\, 2\lambda\big) \ .$$

   Show that under these identifications, the geodesic distance on S^2 (i.e. the angle between two unit vectors) agrees with the Fubini-Studi distance in $\mathbb{P}(\mathbb{C}^2)$.

   Hint: $\big(\cos \tfrac{d}{2}\big)^2 = \tfrac{1}{2}(1 + \cos(d))$.

2. Remind yourself of the definition of a distance (aka. metric) on a set. Show that d is indeed a distance function on $\mathbb{P}(\mathcal{H})$.

   Notes: You may assume that the geodesic distance on the unit-2-sphere satisfies the triangle inequality. (Or you can look up the proof, e.g. in Thm. 2.1.2 of Ratcliffe, Foundations of Hyperbolic Manifolds, Springer.) Try to reduce the general question to the situation in part 1.