Exercise sheet # 04
Topics in representation theory WS 2017

(Ingo Runkel)

Exercise 13

Let G be a connected matrix Lie group (or just a connected topological group), and let $V \subset G$ be any open neighbourhood of the unit $e \in G$. Show that

$$G = \{g_1 \cdots g_n \,|\, n > 0, g_i \in V\} \ .$$

Exercise 14

Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. Let W be a finite-dimensional $\mathbb{C}$-vector space and (W, ρ) a smooth representation of G. Show that for all $X \in \mathfrak{g}$ we have

$$\rho(\exp(X)) = \exp(D\rho(X)) \ .$$

Hint: For $t \in \mathbb{R}$ write $f(t) = \rho(\exp(tX))$ and $g(t) = \exp(tD\rho(X))$. Then f, g are smooth functions $\mathbb{R} \to \mathrm{End}(W)$. Try to find a first order differential equation they both satisfy.

If you are bored: A special case of the above result is that for all $X \in \mathrm{Mat}(N, \mathbb{C})$ we have $\det(\exp(X)) = \exp(\mathrm{tr}(X))$. Do you see how?

Exercise 15

Let $\mathcal{H}$ be a Hilbert space. Remind yourself of the definition of a semidirect product of groups and see if you can write $UA(\mathcal{H})$ as a semidirect product of $U(\mathcal{H})$ with something.

Exercise 16

In the lecture we defined $\gamma : UA(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ as follows. For $X \in UA(\mathcal{H})$ abbreviate $T = \gamma(X)$. Then for $[\varphi] \in \mathbb{P}(\mathcal{H})$ the class of $\varphi \in \mathcal{H}$ we declare $T([\varphi]) = [X\varphi]$.

Show that T is independent of the choice of representative and does indeed lie in $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Show that γ is a group homomorphism.

Please turn over.

Exercise 17

Let L be the Lorentz group of four-dimensional (or n-dimensional, if you like) Minkowski space. Set

$$L_+^\uparrow := \{M \in L \mid \det(M) > 0 \text{ and } M_{00} > 0\} \ .$$

1. Show that $L_+^\uparrow$ is connected.

   Hint: You could show that every element can be written as (rotation) $\times$ (boost) $\times$ (rotation). (You may assume that $SO(3)$ is connected.)

2. How many connected components does L have? Can you write it as a semidirect product of $L_+^\uparrow$ with something?