Exercise sheet # 03
Topics in representation theory WS 2017

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Exercise 9

1. Let k be a field (you can take $k = \mathbb{C}$, it makes no difference) and let A, Q be k-algebras. Suppose there exists a surjective algebra homomorphism $\pi : A \twoheadrightarrow Q$.

   What is the relation between A- and Q-modules, and between their intertwiners? In case A has only finitely many simple modules, what can you say about the number of simple Q-modules?

2. Show that quotients of semisimple algebras are semisimple.

Exercise 10

Show the missing step in Lemma 1.2.14: Given $f_1, \dots, f_n \in \mathrm{End}(V)$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ consider

$$F(\lambda_1, \dots, \lambda_n) = id + \sum_{i=1}^{n} \lambda_i f_i \ .$$

For λ_i small enough, $F(\lambda_1, \dots, \lambda_n) \in GL(V)$. By construction,

$$F(\lambda_1, \dots, \lambda_n)^{\otimes n} \in \mathrm{span}_{\mathbb{C}}\{H^{\otimes n} | H \in GL(V)\}$$

Expanding out $F(\lambda_1, \dots, \lambda_n)$ gives a polynomial in $\lambda_1, \dots, \lambda_n$ with coefficients in $\mathrm{End}(V^{\otimes n})$.

Show that these coefficients are also contained in $\mathrm{span}_{\mathbb{C}}\{H^{\otimes n} | H \in GL(V)\}$.

Hint: A finite dimensional sub-vector space is always closed (as a topological space), and so contains limits of sequences for which all elements are in the sub-vector space.

Exercise 11

Show that $GL(N, \mathbb{C})$, $SL(N, \mathbb{C})$, $U(N)$ and $SU(N)$ are connected.

*H*int: Look up the QR-decomposition of a matrix. For $SU(N)$ proceed by induction by "finding a path which rotates the last column vector to the basis vector e_N."

Please turn over.

Exercise 12

1. Let $D \subset GL(N, \mathbb{C})$ be the subgroup of invertible diagonal matrices. Can the action of D on $(\mathbb{C}^N)^{\otimes n}$ be diagonalised? I.e. is there a basis of $(\mathbb{C}^N)^{\otimes n}$ consisting of simultaneous eigenvectors for all $\rho(D)$? How about the $S_\lambda(\mathbb{C}^N)$?

2. Let $d \subset \mathrm{Mat}(N, \mathbb{C})$ be all diagonal matrices. These are in the Lie algebra of $GL(N, \mathbb{C})$. Compute their action on $(\mathbb{C}^N)^n$. What is the largest eigenvalue the elementary matrix E_{11} can attain on $(\mathbb{C}^N)^{\otimes n}$? Which of the $S_\lambda(\mathbb{C}^N)$, for λ a Young-diagram with n boxes, contain an eigenvector of this maximal eigenvalue?