Exercise sheet # 02
Topics in representation theory WS 2017

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Exercise 5

Let V be an n-dimensional vector space with basis $\{e_1, \dots, e_n\}$.

1. Let $\lambda = (1, 1, \dots, 1)$ be the partition of n into a sum of 1's. I.e. λ is the Young diagram consisting of a column with n boxes.
   (a) Let $\pi := (-).\hat{c}_\lambda$. Show that the image of π is one-dimensional. Set $b := \pi(e_1 \otimes \cdots \otimes e_n)$. What is $\pi(e_{i_1} \otimes \cdots \otimes e_{i_n})$?
   (b) Show that $S_\lambda V$ is isomorphic to the one-dimensional representation where $F \in GL(V)$ acts by multiplication with $\det(F)$.
2. What can you say about $\lambda = (k, k, \dots, k)$, a partition of nk, for $k > 0$?

Exercise 6

Let $\Gamma := \mathbb{C}S_n$ be the group algebra of S_n.

1. Show that $e_g \mapsto e_{g^{-1}}$ is an algebra anti-isomorphism of Γ and that $e_g \mapsto \mathrm{sgn}(g) e_g$ is an algebra isomorphism.
2. Set $\varphi : \Gamma \to \Gamma$, $\varphi(e_g) = \mathrm{sgn}(g) e_{g^{-1}}$. Find a sensible notion of a transpose Young tableau T^t and show $\varphi(a_T) = b_{T^t}$, $\varphi(b_T) = a_{T^t}$.
3. Show that $\varphi(\tilde{V}_T) = V_{T^t}$ as subspaces of Γ. Find a relation between the right action on $\tilde{V}_T$ and the left action on V_{T^t}.
4. Show corresponding statements of Theorem 1.1.4 for the Γ-right modules $\tilde{V}_\lambda$.

Exercise 7

Show Lemma 1.2.8 about commutants. In fact, show a version of this lemma where $\mathrm{End}(W)$ is replaced by an arbitrary algebra E (finite-dimensional or not), and A and B by any subsets (which need not be a sub-vector space). Add a fourth point to the lemma: A' is a subalgebra.

Exercise 8

Let λ, μ be Young-diagrams with $|\lambda| = m$ and $|\mu| = n$. For $\tilde{V}_\lambda$ and $\tilde{V}_\mu$ write $\tilde{V}_\lambda \bullet \tilde{V}_\mu := c_\lambda \otimes c_\mu . \mathbb{C}S_{m+m}$. Here, $c_\lambda \otimes c_\mu$ is understood as an element of $\mathbb{C}S_{m+m}$ via the embedding $S_m \times S_n \to S_{m+n}$. By construction, $\tilde{V}_\lambda \bullet \tilde{V}_\mu$ is a right S_{m+n}-module. (Aside: Can you define $\tilde{V}_\lambda \bullet \tilde{V}_\mu$ using induced modules?)
Show: $S_\lambda V \otimes S_\mu V \cong \mathrm{Hom}_{\mathbb{C}S_{m+m}}(\tilde{V}_\lambda \bullet \tilde{V}_\mu, V^{\otimes(m+n)})$ as $GL(V)$-modules.