Hints and solutions for problem sheet # 08

Advanced Algebra — Winter term 2016/17

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Problem 30

1. Consider the following candidate for a composition series of M:

$$M_i := \begin{cases} g^{-1}(N_i) & 0 \le i \le s \\ f(L_{i-s}) & s \le i \le (s+r) \end{cases}$$

Since g is surjective, $M_0 = g^{-1}(N_0 = N) = M$ and since $L_r = 0$, $M_{s+r} = f(L_r) = 0$. For $i = s$ we have $g^{-1}(N_s) = \ker g = \mathrm{im} f = f(L_0)$.

Simpleness of quotients for the half which is $g^{-1}(N_i)$: if M_{i-1}/M_i not simple for some i, then there's some $\widetilde{M_i}$ with $M_{i-1} \supsetneq \widetilde{M_i} \supsetneq M_i$. Modding all out by $f(L)$ preserves the inclusion order, so that would mean that $N_{i-1} \supsetneq g(\widetilde{M_i}) \supsetneq N_i$. But N_{i-1}/N_i is simple so this cannot happen.

Simpleness of quotients for the half which is $f(L_i)$: That $f(L_{i-1})/f(L_i)$ is simple is clear as f is injective.

We have $l(M) = l(L) + l(N)$ by uniqueness of composition series length and the above construction of the composition series.

2. Start with $0 \to M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$. Then can break up with an exact triangle

$$\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r, "f_0"] & M_1 \arrow[rr, "f_1"] \arrow[dr] & & M_2 \arrow[r, "f_2"] & M_3 \arrow[r] & 0 \\
& & & M_1/\mathrm{im}(f_0) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

and using part 1 applied to the two SESes, have

$$\begin{aligned} l(M_1) &= l(M_0) + l(M_1/\mathrm{im}(f_0)) \\ l(M_2) &= l(M_1/\mathrm{im}(f_0)) + l(M_3) \end{aligned}$$

Thus $l(M_0) - l(M_1) + l(M_2) - l(M_3) = 0$.

Turning this into an induction on n gives $\sum_i (-1)^i l(M_i) = 0$ (Details?).

Problem 31

1. Counter example 1: Let p be a prime number and take $M = \mathbb{Z}$ as a $\mathbb{Z}$-module. Set $M_n = p^n\mathbb{Z}$, $n = 0, 1, 2, \ldots$. Then $M_0 = M$, $M_n \supset M_{n+1}$ and $\bigcap_{n=0}^{\infty} M_n = \{0\}$. Furthermore, $M_n/M_{n+1} \cong \mathbb{Z}/p\mathbb{Z}$, which is simple. Different choices of p now give inequivalent "half-infinite composition series".

   Counter example 2: Consider $M = \mathbb{C}[X]$ as a $\mathbb{C}[X]$-module and pick $\lambda \in \mathbb{C}$. Set $M_n = \langle (X-\lambda)^n \rangle$, $n = 0, 1, 2, \ldots$. Then $M_0 = M$, $M_n \supset M_{n+1}$ and $\bigcap_{n=0}^{\infty} M_n = \{0\}$. Have $M_n/M_{n+1} \cong \mathbb{C}_\lambda$ (notation as in Problem 7).

2. Actually, this generalisation is true. Here is a sketch of the proof.

   Let $\{0\} = M_0 \subset M_1 \subset M_2 \subset \ldots$ and $\{0\} = N_0 \subset N_1 \subset N_2 \subset \ldots$ satisfy the conditions in generalisation 2.

   Claim: For each $i = 0, 1, \ldots$ there is a $j(i)$ such that $M_i \subset N_{j(i)}$.

   Proof: By induction. Pick $x \in M_1$, $x \neq 0$ (by assumption M_1 is simple, hence non-zero). There is $j(1)$ such that $x \in N_{j(1)}$. Then $M_1 \cap N_{j(1)}$ is a non-zero submodule of M_1. But M_1 is simple, hence $M_1 \cap N_{j(1)} = M_1$. For the induction step, repeat the above argument for the chains $M_1/M_1 \subset M_2/M_1 \subset M_3/M_1 \subset \ldots$ and $N_{j(1)}/M_1 \subset N_{j(1)+1}/M_1 \subset N_{j(1)+2}/M_1 \subset \ldots$ This gives $j(2) > j(1)$ such that $M_2/M_1 \subset N_{j(2)}/M_1$, i.e. $M_2 \subset N_{j(2)}$. Etc.

   Fix some $K > 0$. By the common refinement lemma, each simple successive quotient of the chain $\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_K \subset N_{j(K)}$ (where only the last quotient is potentially non-simple) has to occur – with the same or greater multiplicity – in $\{0\} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{j(K)}$ (where every quotient is simple).

   Fix a simple R-module S. From the above observation one concludes that the number of quotients M_{i+1}/M_i that are isomorphic to S (which may be finite or infinite) is smaller or equal to the number of quotients N_{i+1}/N_i isomorphic to S. Exchanging the roles of M and N gives the equality.

Problem 32

1. Consider the sequence $I \cap J \to I \oplus J \to R$, where the first map sends x to $(x, -x)$ and the second sends (y, z) to $y + z$. This is a short exact sequence (why?). It splits since R is a free R-Module (using Proposition 2.4.9). Hence $(I \cap J) \oplus R \cong I \oplus J$.

2a. We will show that I is not principal, the argument for J is the same.

   Note that $|r|^2 = a^2 + 5b^2$ for $r = a + b\sqrt{-5}$. This is a non-negative integer. The two generators of I have norm-squared $|3|^2 = 9$, $|2+\sqrt{-5}|^2 = 9$.

   Let $x := r \cdot 3 + s \cdot (2+\sqrt{-5})$ be an arbitrary element of I (where $r, s \in R$). Then $|x| = |3r + (2+\sqrt{-5})s| \geq \big||3r| - |(2+\sqrt{-5})s|\big| = 3\big||r| - |s|\big|$. But $r, s \in R$ and the points in R form a regular lattice where the smallest distance between any two distinct lattice points is 1. Thus $|x|^2$ is either zero or ≥ 9.

Suppose there is a t such that $\langle t \rangle = I$. Then there are r, s with $rt = 3$ and $st = 2 + \sqrt{-5}$. Since $t \in I$, it must have $|t|^2 = 9$ for this to be possible. But then $|r| = 1 = |s|$, and so $r, s = \pm 1$, which cannot be.

2b. $I + J = R$: $I + J$ contains 3 and 4, and therefore also 1. Non-isomorphic: R is principal, I, J are not.

Problem 33

1. Assume $f(x) \neq 0$. Recall the definition of a function $f : A \to \mathbb{R}$ being continuous at a point x: $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. Let $\epsilon < \frac{1}{2}|f(x)|$. Suppose that for all other $c \in \mathbb{Q}$, we had $f(c) = 0$. Since for each $\delta > 0$ there is a $c \neq x$ with $|x - c| < \delta$ we would then find $|f(x) - f(c)| = |f(x)| > \epsilon$. This is a contradiction to f being continuous.

2. $\mathbb{Q}$ is considered here with the subspace topology; we know what open sets in $\mathbb{R}$ look like ((x, y) and unions and finite intersections), then $\mathcal{O}$ is open in $\mathbb{Q}$ iff $\exists U$ open in $\mathbb{R}$ such that $\mathcal{O} = U \cap \mathbb{Q}$.

   Given $a \in \mathbb{R} - \mathbb{Q}$, $(-\infty, a)$ is an open subset of $\mathbb{R}$ and thus $(-\infty, a) \cap \mathbb{Q}$ is open in $\mathbb{Q}$.

   Now, to show its complement is also open in $\mathbb{Q}$ (i.e. that it's closed in $\mathbb{Q}$): The complement in $\mathbb{R}$ is $[a, +\infty)$, and, being a subspace, the complement in $\mathbb{Q}$ will be its complement in $\mathbb{R}$ then intersect with $\mathbb{Q}$. Since $a, \notin \mathbb{Q}$, we have that $[a, +\infty) \cap \mathbb{Q} = (a, +\infty) \cap \mathbb{Q}$, which is clearly of the form (open set in $\mathbb{R}$) $\cap \mathbb{Q}$.

3. Consider a nonzero ideal M of ${}_R R$. Let $f \in R$ be nonzero. By (a), it has at least two points (say x, y) at which it is nonzero. Let a be an irrational number between these two ($x < a < y$).

   Using $\chi_{U_{<a}}$ and $\chi_{U_{>a}}$, we will construct two submodules of M such that M is their direct sum. (Since this can be done for any M, there are no irreducibles).

   Let $M_1 := \{\chi_{U_{<a}} \cdot m | m \in M\}$ and $M_2 := \{\chi_{U_{>a}} \cdot m | m \in M\}$. These are non-zero submodules of M (why?).

   Since $\chi_{U_{<a}} + \chi_{U_{>a}} = 1$ we have $M_1 + M_2 = M$. To show this is a direct sum, need to show that $M_1 \cap M_2 = 0$

   Consider $m \in M_1 \cap M_2$. There are then elements $m_i \in M_i$ such that $m = \chi_{U_{<a}} \cdot m_1 = \chi_{U_{>a}} \cdot m_2$. Thus $m = \chi_{U_{<a}} \cdot m_1 = \chi_{U_{<a}} \chi_{U_{<a}} \cdot m_1 = \chi_{U_{<a}} \chi_{U_{>a}} \cdot m_2 = 0$.

Problem 34

1. In a composition series, the dimension (over $\mathbb{C}$) of successive quotients is 1, hence the length of the composition series of M coincides with the dimension over $\mathbb{C}$ of M.

   A $\mathbb{C}[X]$-module is the same thing as a $\mathbb{C}$-vector space V together with a choice of endomorphism $f \in \mathrm{End}(V)$. Two $\mathbb{C}[X]$-modules (V, f) and (W, g) are isomorphic if and only if there is a linear isomorphism $\phi : V \to W$ such that $f = \phi^{-1} \circ g \circ \phi$.

   Thus we need to classify pairs $(\mathbb{C}^n, f)$ up to conjugacy and find a condition such that the corresponding module is indecomposable. The classification up to conjugacy is achieved by the Jordan normal form. If there is more than one Jordan cell, there is a non-trivial direct sum decomposition. Since the Jordan normal form is unique up to permutation of cells, if there is only one cell, there cannot be a non-trivial direct sum decomposition.

   We conclude that finite length indecomposable $\mathbb{C}[X]$-modules are classified up to isomorphism by pairs (n, λ), where $n > 0$ gives the dimension over $\mathbb{C}$ and $\lambda \in \mathbb{C}$ gives the generalised eigenvalue of the Jordan cell.

2. The indecomposable module $(\mathbb{C}^n, J(\lambda))$, with $J(\lambda)$ a rank-n Jordan cell for generalised eigenvalue λ has an obvious filtration $\mathbb{C}^n \supset \mathbb{C}^{n-1} \supset \mathbb{C}^{n-2} \supset \dots$ by invariant subspaces. The successive quotients are isomorphic to $\mathbb{C}_\lambda$ (Problem 29).