Hints and solutions for problem sheet # 04
Advanced Algebra — Winter term 2016/17
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Problem 12

1. Let $f : A \to B$ be a morphism in $\mathcal{C}$.

   f is epi in $\mathcal{C}^{\text{op}}$: For all $g, g' : A \to X$ in $\mathcal{C}^{\text{op}}$: $g \circ^{\text{op}} f = g' \circ^{\text{op}} f$ implies $g = g'$.

   Inserting the definition of $\mathcal{C}^{\text{op}}$ to rewrite this in $\mathcal{C}$ gives: For all $g, g' : X \to A$ in $\mathcal{C}$: $f \circ g = f \circ g'$ implies $g = g'$.

   But this is the definition of f being mono in $\mathcal{C}$.

2. (cokernel) Let $f : A \to B$ be a morphism in $\mathcal{C}$, or, equivalently, $f : B \to A$ a morphism in $\mathcal{C}^{\text{op}}$.

   That $(K, k : K \to B)$ is a kernel of f in $\mathcal{C}^{\text{op}}$ means: for all $x : X \to B$ such that $f \circ^{\text{op}} x = 0$ there exists a unique $\tilde{x} : X \to K$ in $\mathcal{C}$ such that $x = k \circ^{\text{op}} \tilde{x}$.

   Rewrite in $\mathcal{C}$: for all $x : B \to X$ such that $x \circ f = 0$ there exists a unique $\tilde{x} : K \to X$ in $\mathcal{C}$ such that $x = \tilde{x} \circ k$.

   But this is the defining property turning $(K, k : B \to K)$ into a cokernel of f in $\mathcal{C}$.

   (coproduct) Let X and $(x_i : X \to X_i)$ be a product of the $(X_i)_{i \in I}$ in $\mathcal{C}^{\text{op}}$.

   Defining property in $\mathcal{C}^{\text{op}}$: For all M, $f_i : M \to X_i$ there exists a unique $f : M \to X$ such that $f_i = x_i \circ^{\text{op}} f$ for all $i \in I$.

   Rewrite in $\mathcal{C}$: For all M, $f_i : X_i \to M$ there exists a unique $f : X \to M$ such that $f_i = f \circ x_i$ for all $i \in I$.

   This is the universal property turning $(X, x_i : X_i \to X)$ into a coproduct of the X_i in $\mathcal{C}$.

Problem 13

1. All properties follow from two observations: i) there is at most one morphism between any two objects, and ii) for $A \neq B$, if there is a morphism $A \to B$ there is no morphism $B \to A$.

2. R be a ring and $\varphi : \mathbb{Q} \to R$ a ring homomorphism. Consider $\frac{p}{q} \in \mathbb{Q}$. Since $q \cdot \frac{1}{q} = 1$, $\varphi(\frac{1}{q})$ is inverse to $\varphi(q)$. Thus $\varphi(\frac{p}{q}) = \varphi(p)\varphi(q)^{-1}$, which shows that φ is uniquely determined by its value on $\mathbb{Z} \subset \mathbb{Q}$.

3. Kernel: We have $K \xrightarrow{k} M \xrightarrow{f} N$ with the factorization property that for any $h : U \to M$ with $f \circ h = 0$, there's a unique map $\tilde{h}$ forcing h to factor over K.

Now assume we have maps $g, g' : U \to K$ such that $k \circ g = k \circ g'$. In particular, $f \circ k \circ g = f \circ k \circ g' = 0$. So, we can draw our diagram as

$$\begin{tikzcd} K \arrow[r, "k"] & M \arrow[r, "f"] & N \\ & U \arrow[u, shift left, "k\circ g'"'] \arrow[u, shift right, "k\circ g"] & \end{tikzcd}$$

Now we're done by uniqueness. (why?)

Cokernel is dual.

Problem 14

1. We clearly get maps $e_i : S_i \to \coprod_{i\in I} S_i$ just as inclusions into the ith coordinate. The disjoint union is such that any element of it must live in one of the factors; $s \in \coprod_I S_i$ can be also viewed as $(s, j) \in S_j$. Given a set V and maps $v_i : S_i \to V$ for all $i \in I$, we need a unique map out of the coproduct factoring over the embeddings. Define $v : \coprod_{i\in I} S_i \to V$ to take (s, j) to $v_j(s)$. This clearly factors as needed, and is unique.

2. Let $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/3\mathbb{Z}$ and denote by S_3 the permutation group of three elements. There are injective group homomorphisms $f_1 : A \to S_3$ and $f_2 : B \to S_3$.

   But the cartesian direct product $A \times B$ is again abelian, and there is no group homomorphism $A \times B \to S_3$ which is injective when restricted to A or B (why?).

   Aside: For a finite collection $(G_i)_{i\in I}$ of groups, abelian or not, and containing more than one non-trivial group, the cartesian product $\prod_i G_i$ is never a coproduct in **Grp**. One can verify this by replacing S_3 in the above argument by the so-called free product of the G_i (which, incidentally, is the coproduct in **Grp**).

Problem 15

The map cannot be an isomorphism because there is a nonzero kernel.

(mono)Assume that we have maps $f, g : A \to \mathbb{Q}$ (keeping in mind that A is a divisible group), such that $\pi \circ f = \pi \circ g$. Since we're in abelian groups, we can add and subtract maps and our equality is equivalent to saying that $\pi(f - g) = 0$. To consider the kernel, we need to work in the larger category Ab for a bit, as $\mathbb{Z}$ is clearly the kernel of π, but not divisible.
$\mathbb{Z}$ being the kernel (via $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$) means that $\pi(f - g) = 0$ implies that we have a map $h : A \to \mathbb{Z}$ such that $f - g = h \circ i$. *But* there are no non-trivial

group homs from a divisible group to $\mathbb{Z}$ (why?); h must be 0, so $f - g = 0$, done.

(epi) Assume that we have maps $f, g : \mathbb{Q}/\mathbb{Z} \to B$ for B some divisible group, with $f \circ \pi = g \circ \pi$. If $f \neq g$, then there must be at least one element $q \in \mathbb{Q}/\mathbb{Z}$ such that $f(q) \neq g(q)$. For every $q \in \mathbb{Q}/\mathbb{Z}$, since π is surjective, there is (at least one) pre-image $\hat{q} \in \mathbb{Q}$ with $\pi(\hat{q}) = q$. Then $f \circ \pi(\hat{q}) = g \circ \pi(\hat{q})$ i.e. $f(q) = g(q)$. So, there are no elements where the two maps can differ; $f = g$.

Problem 16

The missing diagram is not a commuting diagram, as it forms a look (so there is no starting point and no endpoint for which to compare two different composition paths).

In the second case we may take X to be the zero-object $X = \{0\}$ and $x_i = 0$ for all i. Clearly, this satisfies the second property, and so in particular (X, x_i) is unique up to unique isomorphism (by the general argument of uniqueness of solutions to universal properties – repeat the argument to see why).

The first case is not actually a universal property, because (X, x_i) cannot take the role of (M, f_i) (the arrows go the wrong way). In particular, the argument of uniqueness of the solution to a universal property does not apply (why?). And indeed, there is more than one solution: One can set $X = \{0\}$ and $x_i = 0$ for all i. Or one can take X arbitrary and just set x_i to zero (Why does that work?). If I has only one element, say $I = \{1\}$, we can also take $X = X_1$ and $x_1 = id$.