Hints and solutions for problem sheet # 01
Advanced Algebra — Winter term 2016/17
(Ingo Runkel)

___

Problem 1

1. Write $f = \sum_m r_m \chi_m$ and $g = \sum_n s_n \chi_n$ with $r_m, s_n \in R$. By definition $f * g = \sum_{a,b} r_a s_b \chi_{a \cdot b}$, so that

$$(f * g)(m) = \sum_{a,b \in M} r_a s_b \chi_{a \cdot b}(m) = \sum_{a,b \in M, a \cdot b = m} r_a s_b \ .$$

   But $r_a = f(a)$ and $s_b = g(b)$.

2. That $M[R]$ is an abelian group with respect to addition and that the distributive law holds are immediate. For associativity of the product, note that

$$(f * (g * h))(m) = \sum_{a,b \in M, a \cdot b = m} f(a)(g * h)(b)$$
$$= \sum_{a,b \in M, a \cdot b = m} \sum_{c,d \in M, c \cdot d = b} f(a)g(c)h(d)$$
$$= \sum_{a,c,d \in M, a \cdot (c \cdot d) = m} f(a)g(c)h(d) \ .$$

   The product $(f * g) * h$ gives the same result.

   The unit is given by χ_1, i.e. the characteristic function of the unit of the monoid M.

3. The isomorphism is given by $\sum_m r_m \chi_m \mapsto \sum_m r_m X^m$.

4. The element $\chi_1 + \chi_{-1}$ is a non-zero zero-divisor (in particular, it has no multiplicative inverse). Namely,

$$(\chi_1 + \chi_{-1})(\chi_1 - \chi_{-1}) = \chi_1 \chi_1 - \chi_{-1} \chi_{-1} = \chi_{1 \cdot 1} - \chi_{(-1) \cdot (-1)} = \chi_1 - \chi_1 = 0 \ .$$

Problem 2

By Problem 3, giving an $R[X_1, \ldots, X_n]$-module is the same as giving a ring homomorphism $f : R[X_1, \ldots, X_n] \to \mathrm{End}(M)$.

Suppose we are given f as above. Then $\alpha_i := f(X_i) \in \mathrm{End}(M)$ obeys $\alpha_i \alpha_j = f(X_i)f(X_j) = f(X_i X_j) = f(X_j X_i) = f(X_j)f(X_i) = \alpha_j \alpha_i$. For all $r \in R$, $p \in R[X_1, \ldots, X_n]$ and $m \in M$ we have $\alpha_i(r.m) = f(X_i)(r.m) = X_i.(r.m) = (X_i r).m = (r X_i).m = r.(X_i.m) = r.f(X_i)(m) = r.(\alpha_i(m))$. Thus $\alpha_i \in \mathrm{End}_R(M)$.

Conversely, let α_i be as in the problem. Define f on generators as $f(X_i) := \alpha_i$. That is, on a general element we have

$$f(\sum u_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}) = \sum u_{i_1,\ldots,i_n} \alpha_1^{i_1} \cdots \alpha_n^{i_n} \ .$$

One checks that this is an ring homomorphism to $\text{End}(M)$ and that its image is in fact in $\text{End}_R(M)$.

Problem 3

1. For $\sigma \in A$ set $F(\sigma) : R \to \text{End}(M)$ to be $F(\sigma)(r) : m \mapsto \sigma(r,m)$. One checks that $F(\sigma) \in B$. For example, compatibility with the product is established by (abbreviate $f = F(\sigma)$)

$$f(rs)(m) = \sigma(rs,m) = \sigma(r,\sigma(s,m)) = f(r)(\sigma(s,m)) = f(r)(f(s)(m)) \ ,$$

i.e. $f(rs) = f(r)f(s)$. (Details for other properties?)

Conversely, for $\rho \in B$ set $G(\rho) : R \times M \to M$ to be $G(\rho) : (r,m) \mapsto (\rho(r))(m)$. As above one checks that $G(\rho) \in A$ (details?).

Furthermore, one verifies that A and B are each otheres inverse. For example, for all $r \in R$, $m \in M$,

$$(F(G(\rho))(r))(m) = G(\rho)(r,m) = \rho(r)(m) \ ,$$

that is, $F(G(\rho)) = \rho$. (Details for other order of composition?)

2. **Definition.** Let (M,ρ), (N,τ) be representations of R. A *homomorphism of representations* from M to N is a map $f : M \to N$ which is a homomorphism of abelian groups such that in addition, for each $r \in R$,

$$f \circ \rho(r) = \tau(r) \circ f \ .$$

Let M, N be as in the above definition. Write M' for the R-module obtained via the map G, i.e. $M' = (M, G(\rho))$. Similarly, $N' = (N, G(\tau))$. Let $f : M \to N$ be a map (of sets, no further properties for now).

Claim. f is a homomorphism of representations from M to N if and only if it is an R-module homomorphism from M' to N'.

We show one direction as an example: Suppose f is a homomorphism of representations from M to N. Then it is in particular a homomorphism $M \to N$ of abelian groups. It remains to show that $f(r.m) = r.f(m)$ for all $r \in R$, $m \in M$. Writing out the actions explicitly, this becomes $f(G(\rho)(r,m)) = G(\tau)(r, f(m))$. Indeed,

$$f(G(\rho)(r,m)) = f(\rho(r)(m)) = (f \circ \rho(r))(m) = (\tau(r) \circ f)(m) = G(\tau)(r, f(m)) \ .$$

(Details for other direction?)