Problem sheet # 12 Advanced Algebra
Winter term 2016/17

(Ingo Runkel)

Problem 50 (More properties of E)

1. Show the converse of Remark 5.3.2: If $E(P, M) = 0$ for all M then P is projective; if $E(M, J) = 0$ for all M then J is injective.

2. Let R be a principal ideal domain and let $a \in R$ be non-zero. Show that for every R-module M we have $E(R/Ra, M) \cong M/aM$. Give the isomorphism explicitly.

Problem 51 (Examples of extensions)

Let K be a field. Recall the notation $K_\lambda = K[X]/(X - \lambda)$, $\lambda \in K$ for the simple $K[X]$-modules (Problem 29, Sheet 7).

1. Compute $E(K_\lambda, K_\mu)$.

2. Compute $E(K[X], K_\lambda)$.

3. Compute $E(K_\lambda, K[X])$.

4. Show that in any non-split short exact sequence $K[X] \to B \to K_\lambda$, B is isomorphic to $K[X]$ as a $K[X]$-module.

Problem 52 (Exact functors)

You overheard someone say "To test a functor for left or right exactness, it is enough to test it on short exact sequences." Make that statement precise and prove it.

Problem 53 (Complexes and homologies)

Let $\mathbf{C}$ be a chain complex of free abelian groups.

1. Which of the $\mathbb{Z}$-modules B_n, Z_n, H_n are always free? Show that Z_n is a direct summand of C_n.

2. Suppose $C_n = 0$ for $n < 0$ and for $n > N$, and that all C_n are finitely generated. Let $c_n \in \mathbb{Z}_{\geq 0}$ be the rank of C_n, and h_n the rank of $H_n(\mathbf{C})$. Show that

$$\sum_{n=0}^{N} (-1)^n c_n \;=\; \sum_{n=0}^{N} (-1)^n h_n \;.$$