Problem sheet # 04 Advanced Algebra Winter term 2016/17

(Ingo Runkel)

Problem 12 (Opposite notions)

1. Let f be a morphism in a category $\mathcal{C}$. Show that f is mono in $\mathcal{C}$ iff it is epi in $\mathcal{C}^{\mathrm{op}}$.
2. Often one writes "X is a co(something) in $\mathcal{C}$" if X is a (something) in $\mathcal{C}^{\mathrm{op}}$. Make this precise for "cokernel" or for "coproduct" (or both if you like).

Problem 13 (Mono and epi)
Show:

1. In the category formed by a partial ordered set, every morphism is mono and epi, but only the identity is invertible.
2. In the category of rings, the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epi.
3. In a category with zero object, if the kernel of a morphism f exists, it is mono, and if the cokernel of f exists, it is epi.

Problem 14 (Coproducts)

1. Consider the category **Set** of sets and maps between sets. Show that every family $(S_i)_{i \in I}$ of objects in **Set** has a coproduct, and that this coproduct is given by the disjoint union.
2. Recall that in **Ab**, the coproduct of two groups A, B is the direct sum $A \oplus B$, which for finite index sets (as here) is just the cartesian product $A \times B$.

   Consider the category **Grp** of groups and group-homomorphisms. Let A, B be two abelian groups. Show that $A \times B$ is in general not a coproduct in **Grp**. (It is enough to give one counter-example.)

Problem 15 (Mono, epi, iso)
A *divisible group* is an abelian group A such that for any $a \in A$ and $n \in \mathbb{Z}_{>0}$ there exists a $b \in A$ such that $nb = a$. The category of divisible groups is the full subcategory of **Ab** formed by divisible groups (i.e. its objects are divisible groups and its morphisms group homomorphisms).

Show that the canonical projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is mono and epi in the category of divisible groups, but not an isomorphism.

Please turn over.

Problem 16 (It is not a product)

A product $(X, x_i)_{i\in I}$ of a family $(X_i)_{i\in I}$ in a category $\mathcal{C}$ is characterised by a universal property of the form, for all M, f_i,

$$\begin{tikzcd} M \arrow[rr, dashed, "\exists! f"] \arrow[dr, "f_i"'] & & X \arrow[dl, "x_i"] \\ & X_i & \end{tikzcd}$$

Consider the following variations of this universal property in the example of the category **Ab**:

$$\begin{tikzcd} M \arrow[rr, dashed, "\exists! f"] \arrow[dr, "f_i"'] & & X \\ & X_i \arrow[ur, "x_i"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, dashed, "\exists! f"] & & X \\ & X_i \arrow[ul, "f_i"] \arrow[ur, "x_i"'] & \end{tikzcd}$$

(One arrangement of the bottom two arrows is missing. Why?) What can you say about existence and uniqueness of X in **Ab**?