

# Problem sheet #04 Advanced Algebra

## Winter term 2016/17

(Ingo Runkel)

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### Problem 12 (Opposite notions)

1. Let  $f$  be a morphism in a category  $\mathcal{C}$ . Show that  $f$  is mono in  $\mathcal{C}$  iff it is epi in  $\mathcal{C}^{\text{op}}$ .
2. Often one writes “ $X$  is a co(something) in  $\mathcal{C}$ ” if  $X$  is a (something) in  $\mathcal{C}^{\text{op}}$ . Make this precise for “cokernel” or for “coproduct” (or both if you like).

### Problem 13 (Mono and epi)

Show:

1. In the category formed by a partial ordered set, every morphism is mono and epi, but only the identity is invertible.
2. In the category of rings, the embedding  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  is epi.
3. In a category with zero object, if the kernel of a morphism  $f$  exists, it is mono, and if the cokernel of  $f$  exists, it is epi.

### Problem 14 (Coproducts)

1. Consider the category **Set** of sets and maps between sets. Show that every family  $(S_i)_{i \in I}$  of objects in **Set** has a coproduct, and that this coproduct is given by the disjoint union.
2. Recall that in **Ab**, the coproduct of two groups  $A, B$  is the direct sum  $A \oplus B$ , which for finite index sets (as here) is just the cartesian product  $A \times B$ .  
Consider the category **Grp** of groups and group-homomorphisms. Let  $A, B$  be two abelian groups. Show that  $A \times B$  is in general not a coproduct in **Grp**. (It is enough to give one counter-example.)

### Problem 15 (Mono, epi, iso)

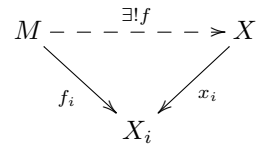
A *divisible group* is an abelian group  $A$  such that for any  $a \in A$  and  $n \in \mathbb{Z}_{>0}$  there exists a  $b \in A$  such that  $nb = a$ . The category of divisible groups is the full subcategory of **Ab** formed by divisible groups (i.e. its objects are divisible groups and its morphisms group homomorphisms).

Show that the canonical projection  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  is mono and epi in the category of divisible groups, but not an isomorphism.

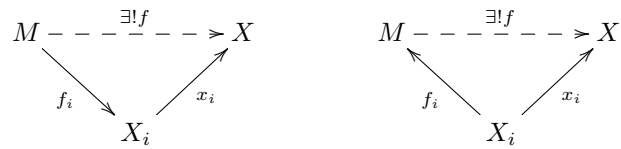
**Please turn over.**

**Problem 16** (It is not a product)

A product  $(X, x_i)_{i \in I}$  of a family  $(X_i)_{i \in I}$  in a category  $\mathcal{C}$  is characterised by a universal property of the form, for all  $M, f_i$ ,



Consider the following variations of this universal property in the example of the category  $\mathbf{Ab}$ :



(One arrangement of the bottom two arrows is missing. Why?) What can you say about existence and uniqueness of  $X$  in  $\mathbf{Ab}$ ?