Reconstruction, Pt. 2

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1 Twistings

This section deals with a method of "twisting" the coproduct of Hopf Algebras to obtain new ones.
Let $(A, \mu, 1, \Delta, \epsilon)$ be a bialgebra over a field k.

Notation 1.1. Recall, that for an element $K = \sum_i s_i \otimes t_i \in A \otimes A$ we set

$$K_{12} = \sum_i s_i \otimes t_i \otimes 1, \quad K_{23} = \sum_i 1 \otimes s_i \otimes t_i, \quad K_{13} \sum_i s_i \otimes 1 \otimes t_i$$

and

$$K_{21} = \sum_i t_i \otimes s_i$$

Let $\mathcal{F} \in A \otimes A$ be invertible, such that

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{F})$$
$$(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathcal{F}).$$

Lemma 1.2. *Let $S \in A$ be an antipode, then $v := \mu(\mathrm{id} \otimes S)(\mathcal{F})$ is invertible with*

$$v^{-1} = \mu(S \otimes \mathrm{id})(\mathcal{F}^{-1}).$$

Proposition 1.3. *$(A, \mu, 1, \Delta^{\mathcal{F}}, \epsilon)$ is a bialgebra, where*

$$\Delta^{\mathcal{F}}(a) := \mathcal{F}\Delta(a)\mathcal{F}^{-1}$$

for $a \in A$. $(A, \mu, 1, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}})$ is a Hopf algebra, where

$$S^{\mathcal{F}}(a) := vS(a)v^{-1}.$$

Both are denoted by $A^{\mathcal{F}}$ and called the <u>twist</u> of A by $\mathcal{F}$.

Notice that it is actually a twist of the coproduct of A.

Corollary 1.4. *If A is cocommutative, then $A^{\mathcal{F}}$ is a triangular bialgebra with universal R-matrix*

$$R = \mathcal{F}_{21}\mathcal{F}^{-1}.$$

Corollary 1.5. *Assume that A is quasitriangular with R-matrix $\mathcal{R}$ and that $\mathcal{F}_{21} = \mathcal{F}^{-1}$ and $\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}$. Then $A^{\mathcal{F}}$ is also quasitriangular with R-matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}^{-1}\mathcal{R}\mathcal{F}^{-1}$.*

Proposition 1.6. $\mathrm{Mod}(A)$ *and* $\mathrm{Mod}(A^{\mathcal{F}})$ *are equivalent as monoidal categories. The equivalence is given by the identity functor with a monoidal structure ϕ, defined as follows:*

$$\phi_{U,V} = (\rho_U \otimes \rho_V)(\mathcal{F}^{-1}),$$

where $\rho_U(a) = (u \mapsto a.u \quad \forall u \in U)$.

This shows that usually several different bialgebras exist, that have the same representation category up to equivalence.

2 Reconstruction of a bialgebra

As it is already implied by the titel, all of the given constructions can be "reversed". We already have an interim result from reconstruction, pt.1:

Proposition 2.1. *Given an algebra A and the forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vec}_k$, $\mathrm{End}(F_A)$ is an algebra with multiplication being the composition and is isomorphic to A via the canonical isomorphism*

$$\rho : A \to \mathrm{End}(F_A).$$

Now assume that A is a bialgebra, giving $\mathrm{Mod}(A)$ a monoidal structure. Furthermore, it can easily be seen that F_A is a monoidal functor with the structural isomorphisms being the identity morphisms for every object.
We can recover the bialgebra structure from this data but some technical problems arise: There is a natural candidate for the coproduct Δ and the counit ϵ given by

$$\Delta(h)_{X,Y} = h_{X\otimes Y}, \quad \epsilon(h) = h_I, \quad \text{for} \quad h \in \mathrm{End}(F_A).$$

It is $\Delta(h) \in \mathrm{End}({F_A}^2)$, where ${F_A}^2 : \mathrm{Mod}(A) \times \mathrm{Mod}(A) \to \mathrm{Vec}_k$ is given by

$$F_A^2(X,Y) := F_A(X) \otimes F_A(Y)$$

and similarly on morphisms. For Δ being a coproduct, we need $\Delta(h) \in \mathrm{End}(F_A) \otimes \mathrm{End}(F_A)$. Therefore, a more formal treatment is necessary. We do this by taking a little indirect route:

Proposition 2.2. $\mathrm{End}(F_A)$ *is a <u>representation</u> of the functor* $\mathrm{Nat}(- \otimes F_A, F_A) : \mathrm{Vec}_k \to \mathrm{Set}$, *i.e. that there is a natural family of isomorphisms*

$$\Theta_V : \mathrm{Lin}(V, \mathrm{End}(F_A)) := \mathrm{Hom}_{\mathrm{Vec}_k}(V, \mathrm{End}(F_A)) \to \mathrm{Nat}(V \otimes F_A, F_A).$$

Here, $V \otimes F_A$ is the functor given by $(V \otimes F_A)(X) = V \otimes F_A(X)$ on objects an by $(V \otimes F_A)(f) = \mathrm{id} \otimes F_A(f)$ on morphisms. We say that $\mathrm{Nat}(- \otimes F_A, F_A)$ is <u>representable</u>.

Θ is given by sending a linear map $\phi : V \to \mathrm{End}(F_A)$ to

$$\Theta(\phi)_X(v \otimes x) = \phi(v)(x) \quad \text{for} \ \ x \in F_A(X).$$

Remark 2.3. Since c and A are isomorphic as vector spaces, we can also choose A as a representation. Correspondingly, Θ then sends a map $\phi : V \to A$ to

$$\Theta(\phi)_X(v \otimes x) = \phi(v).x$$

Θ especially gives us a natural isomorphism

$$(\Theta_{\mathrm{End}(F_A)}(\mathrm{id})_X =: \alpha_X : \mathrm{End}(F_A) \otimes F_A(X) \to F_A(X))_{X \in \mathrm{Ob}(\mathrm{Mod}(A))}.$$

Furthermore:
$\mathrm{Nat}(- \otimes F_A^n, F_A^n)$ can be represented similarly via maps

$$\Theta_V^n : \mathrm{Lin}(V, \mathrm{End}(F_A)^{\otimes n}) \to \mathrm{Nat}(V \otimes F_A^n, F_A^n).$$

We can finally define a coproduct on $\mathrm{End}(F_A)$, or A respectively.

Theorem 2.4. $\mathrm{End}(F_A)$ *with the map given as the inverse image under $\Theta^2_{\mathrm{End}(F_A)}$ of the natural transformation $\alpha_{X\otimes Y} : \mathrm{End}(F_A) \otimes X \otimes Y \to X \otimes Y$ is a bialgebra which is isomorphic to A (with the original coproduct) via ρ.*

3 Abstract reconstruction

In this section we state a reconstruction theorem that does not use the fact that we already have a given representation theory. The general concept of reconstruction is that given an abstract monoidal category $\mathcal{C}$ which satisfies certain conditions together with a "fiber functor", one can think of $\mathcal{C}$ as the representation category of a bialgebra. This can be a powerful tool for studying monoidal categories. Lets begin with clearifying what a fiber functor is.

In an abelian category, there is the notion of an exact sequence:

Definition 3.1. A sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to ...A_n$$

of objects and morphisms in an abelian category is called <u>exact</u> if the image of each morphism is equal to the kernel of the next. A functor $F : \mathcal{P} \to \mathcal{Q}$ between to abelian categories is called <u>exact</u> if for any (short) exact sequence

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is also exact.

Definition 3.2. Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor between two categories. F induces maps

$$F_{X,Y} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$$

for every pair of objects X and Y in $\mathcal{P}$. F is said to be <u>faithful</u> if $F_{X,Y}$ is injective.

Definition 3.3. Let $\mathcal{C}$ be an essentially small, k-linear, abelian monoidal category. A monoidal functor $F : \mathcal{C} \to \mathrm{Vec}_k$ is called a <u>fiber functor</u> if it is k-linear, exact and faithful.

Example 3.4. It is easy to see that the forgetful functor $F_A : \mathrm{Mod}(A) \to \mathrm{Vec}_k$ for a bialgebra A is a fiber functor.

We can give a first statement:

Theorem 3.5. *Let $\mathcal{C}$ be an essentially small, k-linear, abelian, finite monoidal category and let F be a fiber functor $F : \mathcal{C} \to \mathrm{Vec}_k$. Then, it exists a finite dimensional bialgebra A, such that* $\mathrm{Mod}(A)$ *is equivalent to $\mathcal{C}$ as a monoidal category.*

Section 2 provides the strategy for the infinite case:
Let $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ be the structural isomorphisms of the monoidal structure associated to the fiber functor F. Then, there is again a canonical candidate for the comultiplication Δ and the counit ϵ:

$$\Delta(h)_{X,Y} = c^{-1}_{X,Y} \circ h_{X \otimes Y} \circ c_{X,Y}, \quad \epsilon(h) = h_I, \quad \text{for} \quad h \in \mathrm{End}(F).$$

If we assume the existence of a natural family of isomorphisms

$$\Theta^n_V = \Theta^n_V : \mathrm{Lin}(V, A^{\otimes n}) \to \mathrm{Nat}(V \otimes F^n, F^n),$$

A can again be endowed with the structure of a bialgebra.
A more concrete statement is for technical reasons formulated in the dual setting:

Theorem 3.6. *Let $\mathcal{C}$ be an essentially small, k-linear, abelian, monoidal category and let F be a fiber functor $F : \mathcal{C} \to \mathrm{Vec}_k$. Then, it exists a bialgebra A, such that* $\mathrm{Comod}(A)$ *is equivalent to $\mathcal{C}$ as a monoidal category.*

In this setting it is necessary to show the existence of a natural family of isomorphisms

$$\Theta^n_V = \Theta^n_V : \mathrm{Lin}(A^{\otimes n}, V) \to \mathrm{Nat}(F^n, F^n \otimes V).$$

4 Outlook

Theorem 4.1. *Let $\mathcal{C}$ be an essentially small k-linear, abelian, rigid monoidal category with a fiber functor F that maps into the subcategory of finite vector spaces. Then the bialgebra A has an antipode, i.e. it is a Hopf algebra.*

Definition 4.2. A braided monoidal category is a monoidal category $\mathcal{C}$ together with a braiding, which is a choice of a natural isomorphism of the form $\psi_{A,B} : A \otimes B \to B \otimes A$ for each pair of objects A and B, such that it satisfies the following to axioms:

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\psi"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "\psi \otimes I"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[ur, "I \otimes \psi"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\psi"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "I \otimes \psi"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"'] & (A \otimes C) \otimes B \arrow[ur, "\psi \otimes I"'] &
\end{tikzcd}$$

Here α is the associativity isomorphism coming from the monoidal structure on $\mathcal{C}$.

Theorem 4.3. *Let $\mathcal{C}$ be a k-linear, abelian, braided monoidal category and F a fiber functor. Then, A is a quasitriangular bialgebra.*

Remark 4.4. Many examples of braided monoidal categories arise from two-dimensional conformal quantum field theories. These categories can often be obtained from quantum groups as well. This is called Kazhdan-Lusztig correspondence and if I am lucky it will make my master thesis much easier.

References

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