1 R-Matrix For A Quotient Of $U_q(\mathfrak{sl}(2))$

As in the last talk we take the underlying field to be $\mathbb{C}$ and let $q \in \mathbb{C}$ be a root of unity of order d. We assume d is an odd integer with $d > 1$.

1.1 Quotient Of $U_q(\mathfrak{sl}(2))$

Recall the Hopf Algebra $U_q := U_q(\mathfrak{sl}(2))$ generated by E, F, K, K^{-1}.

Proposition 1.1. E^d, F^d, K^d are central.

Let I be the (two-sided) ideal

$$I = \left(E^d, F^d, K^d - 1\right) \tag{1.1}$$

We define the quotient algebra $\bar{U}_q := U_q/I$. $\bar{U}_q$ is finite-dimensional. In particular,

Proposition 1.2. The set $\{E^i F^j K^l\}_{0 \le i,j,l \le d-1}$ is a basis of $\bar{U}_q$

We will construct a universal R-matrix for $\bar{U}_q$ using the quantum double construction.

Proposition 1.3. There exists a unique Hopf Algebra structure on $\bar{U}_q$ such that the canonical projection $\pi : U_q \to \bar{U}_q$ is a morphism of Hopf Algebras.

Proof. We know that given a Hopf Algebra H and a Hopf Ideal I of H there exists a unique Hopf Algebra structure on H/I such that the canonical projection $\pi : H \to H/I$ is a morphism of Hopf Algebras. It then remains to check that I is a Hopf Ideal. We can check

$$\begin{aligned}
\Delta(E)^d = \Delta(F)^d = \Delta(K)^d - 1 = 0 \\
\epsilon(E)^d = \epsilon(F)^d = \epsilon(K)^d - 1 = 0 \\
S(E)^d = S(F)^d = S(K)^d - 1 = 0
\end{aligned} \tag{1.2}$$

□

Define B_q as the subspace of $\bar{U}_q$ spanned by the set $\{E^m K^n\}_{0 \le m,n \le d-1}$.

Proposition 1.4. B_q is a Hopf subalgebra of $\bar{U}_q$.

We now apply the quantum double construction to obtain the quantum double $D(B_q)$ of B_q.

First we need to determine $X = (B_q^{op})^*$ as a Hopf Algebra.

Lemma 1.1. Let $\alpha, \eta \in B_q^*$ defined by

$$\langle \alpha, E^m K^n \rangle = \delta_{m0} q^{2n}, \quad \langle \eta, E^m K^n \rangle = \delta_{m1}. \tag{1.3}$$

Then $\{\eta^i \alpha^j\}_{0 \le i,j \le d-1}$ is a basis of X and X is a Hopf Algebra with

$$\alpha^d = 1, \quad \eta^d = 0, \quad \alpha\eta\alpha^{-1} = q^{-2}\eta,$$

$$\Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(\eta) = 1 \otimes \eta + \eta \otimes \alpha,$$

$$\epsilon(\alpha) = 1, \quad \epsilon(\eta) = 0,$$

$$S(\alpha) = \alpha^{d-1}, \quad S(\eta) = -\eta\alpha^{d-1}$$

Lemma 1.2. The following relations hold in $D = D(B_q)$

$$K\alpha = \alpha K, \quad K\eta = q^{-2}\eta K$$

$$E\alpha = q^{-2}\alpha E, \quad E\eta = -q^{-2}(1 - \eta E - \alpha K) \tag{1.4}$$

In what follows we denote $\eta^i \alpha^j \otimes E^k K^l = \eta^i \alpha^j E^k K^l$.

Proposition 1.5. The linear map $\chi : D(B_q) \to \bar{U}_q$ determined by

$$\chi(\eta^i \alpha^j E^k K^l) = \left(\frac{q - q^{-1}}{q^2} \right)^i q^{2(i+j)k - i(i-1)} F^i E^k K^{i+j+l} \tag{1.5}$$

with $0 \le i, j, k, l \le d - 1$ is a surjective Hopf Algebra morphism.

Proof. Surjectivity is clear since the image of $\{\eta^i \alpha^j E^k K^l\}$ generates $\bar{U}_q$. We first need to show that χ is an algebra morphism. It is enough to show that the images of the generators satisfy the above relations, eg that $\chi(K)\chi(\alpha) = \chi(\alpha)\chi(K)$. Similarly, to show that χ respects the comultiplication and antipode it is enough to check it on the generators. $\square$

Corollary 1.1. The Hopf Algebra $\bar{U}_q$ is quasi-triangular.

Proof. We know that $D = D(B_q)$ is quasi triangular. Let $R_D \in D \otimes D$ be its universal R-matrix. Define $\bar{R} \in \bar{U}_q \otimes \bar{U}_q$ by

$$\bar{R} = (\chi \otimes \chi)(R_D) \tag{1.6}$$

$\bar{R}$ is invertible and since χ is a surjective Hopf Algebra morphism it follows that $\bar{U}_q$ is quasi-triangular. $\square$

The following is our main result:

Theorem 1. *The universal R-Matrix $\bar{R}$ of $\bar{U}_q$ is given by*

$$\bar{R} = \frac{1}{d} \sum_{0 \le i,j,k \le d-1} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k K^i \otimes F^k K^j \tag{1.7}$$

Proof. We will prove the general form of $\bar{R}$. Given a basis $\{e_i\}_{i \in I}$ of B_q with corresponding dual basis $\{e^i\}_{i \in I}$ we have

$$R_D = \sum_{i \in I} e_i \otimes e^i \tag{1.8}$$

Hence

$$\bar{R} = (\chi \otimes \chi)(R_D) = \sum_{i \in I} \chi(e_i) \otimes \chi(e^i) \tag{1.9}$$

We know $\{E^i K^j\}_{0 \le i,j \le d-1}$ is a basis of B_q. Denote by $\{\beta^{ij}\}_{0 \le i,j \le d-1}$ the corresponding dual basis which can be expanded as

$$\beta^{ij} = \sum_{0 \le k,l \le d-1} \mu^{ij}_{kl} \eta^k \alpha^l \tag{1.10}$$

for some coefficients μ^{ij}_{kl}. One can show that $\mu^{ij}_{kl} = 0$ for $i \neq k$. Hence

$$\bar{R} = \sum_{i \in I} \mu^{ij}_{il} \chi(E^i K^j) \otimes \chi(\eta^i \alpha^l) \tag{1.11}$$

Now using our explicit formula for χ we obtain

$$\bar{R} = \sum_{0 \le i,j,k \le d-1} c_{ijk} E^k K^i \otimes F^k K^j \tag{1.12}$$

for come coefficients c_{ijk}. □

1.2 R-Matrix on $V_1 \otimes V_1$

Let V_1 be the vector space spanned by $\{v_0, v_1\}$. The following defines a representation of $\bar{U}_q$ on V_1

$$\begin{aligned} K v_0 &= q v_0, \quad K v_1 = q^{-1} v_1 \\ E v_0 &= 0, \quad E v_1 = v_0 \\ F v_0 &= v_1, \quad F v_1 = 0 \end{aligned} \tag{1.13}$$

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Recall that given a universal R-matrix R for a Hopf Algebra H and finite dimensional H-modules V and W then a solution $c^R_{V,W}$ of the Yang-Baxter Equation (YBE) is given by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)), \quad v \in V, w \in W \tag{1.14}$$

Using the R-matrix given above and the module V_1 of $\bar{U}_q$ then a solution of the YBE is given by

$$\begin{aligned} c^{\bar{R}}_{V_1,V_1}(v_0 \otimes v_0) &= \lambda q v_0 \otimes v_0 \\ c^{\bar{R}}_{V_1,V_1}(v_0 \otimes v_1) &= \lambda v_1 \otimes v_0 \\ c^{\bar{R}}_{V_1,V_1}(v_1 \otimes v_0) &= \lambda(v_0 \otimes v_1 + (q + q^{-1})v_1 \otimes v_0) \\ c^{\bar{R}}_{V_1,V_1}(v_1 \otimes v_1) &= \lambda q v_1 \otimes v_1 \end{aligned} \tag{1.15}$$

where $\lambda = q^{(d-1)/2}$.

We know that $\{v_0, v_1\}$ is a basis of V_1, hence $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ is a basis of $V_1 \otimes V_1$. With respect to this basis the R-matrix takes the form

$$c^{\bar{R}}_{V_1,V_1} = \begin{pmatrix} \lambda q & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & \lambda(q - q^{-1}) & 0 \\ 0 & 0 & 0 & \lambda q \end{pmatrix} \tag{1.16}$$