8 The quantum double

From now on, all algebras $\mathcal{A}$ are finite-dimensional.

Definition 8.1. A dual pairing of two bialgebras $\mathcal{U}$ and $\mathcal{A}$ is a bilinear mapping $\langle .,. \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{K}$, such that

$$\langle f, a_1 a_2 \rangle = \langle \Delta_{\mathcal{U}}(f), a_1 \otimes a_2 \rangle, \ \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta_{\mathcal{A}}(a) \rangle$$
$$\langle f, 1_{\mathcal{A}} \rangle = \epsilon_{\mathcal{U}}(f), \ \langle 1_{\mathcal{U}}, a \rangle = \epsilon_{\mathcal{A}}(a)$$

for all $f, f_1, f_2 \in \mathcal{U}, a, a_1, a_2 \in \mathcal{A}$. Note that the pairing of $\mathcal{U}$ and $\mathcal{A}$ must be extended to one of $\mathcal{U} \otimes \mathcal{U}$ and $\mathcal{A} \otimes \mathcal{A}$ by setting $\langle f_1 \otimes f_2, a \otimes b \rangle := \langle f_1, a \rangle \langle f_2, b \rangle$.

Definition 8.2. A bilinear map $\sigma : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ is called a **skew-pairing** of the bialgebras $\mathcal{A}$ and $\mathcal{B}$ if $\sigma(\cdot, \cdot)$ is a dual pairing of the bialgebras $\mathcal{A}$ and $\mathcal{B}^{op}$, that is, for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ we have

$$\sigma(a, 1) = \epsilon(a), \ \sigma(1, b) = \epsilon(b)$$
$$\sigma(a_1 a_2, b) = \sum_{(b)} \sigma(a_1, b^{(1)}) \sigma(a_2, b^{(2)})$$
$$\sigma(a, b_1 b_2) = \sum_{(a)} \sigma(a^{(2)}, b_1) \sigma(a^{(1)}, b_2)$$

The mapping σ is said to be convolution invertible (or briefly, invertible), if there exists another bilinear map $\overline{\sigma} : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ such that $\sigma \overline{\sigma} = \overline{\sigma} \sigma = \epsilon_{\mathcal{A}} \otimes \epsilon_{\mathcal{B}}$. In Sweedler Notation we get:

$$\sum_{(a),(b)} \sigma(a^{(1)}, b^{(1)}) \overline{\sigma}(a^{(2)}, b^{(2)}) = \sum_{(a),(b)} \overline{\sigma}(a^{(1)}, b^{(1)}) \sigma(a^{(2)}, b^{(2)}) = \epsilon_{\mathcal{A}}(a) \epsilon_{\mathcal{B}}(b)$$

The inverse of σ is denoted by $\overline{\sigma}$. Define $\sigma_{21}(b, a) = \sigma(a, b)$. It is easily seen, that σ is a skew-pairing of $\mathcal{A}$ and $\mathcal{B}$ iff $\overline{\sigma}_{21}$ is a skew-pairing of $\mathcal{B}$ and $\mathcal{A}$, where $\overline{\sigma}_{21}(b, a) = \overline{\sigma}(a, b)$

If either $\mathcal{A}$ or $\mathcal{B}$ is a Hopf algebra with invertible antipode then the skew-pairing of $\mathcal{A}$ and $\mathcal{B}$ is invertible. The inverse is then given by

$$\overline{\sigma} = \sigma(S(a), b) \text{ resp. } \overline{\sigma}(a, b) = \sigma(a, S^{-1}(b)), a \in \mathcal{A}, b \in \mathcal{B} \qquad (1)$$

Proposition 8.3. *i) Let $\mathcal{A}$ and $\mathcal{B}$ be bialgebras equipped with an invertible scew-pairing $\sigma : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$. Then the vector space $\mathcal{B} \otimes \mathcal{A}$ becomes* *an algebra with product defined by*

$$(b_1 \otimes a_1)(b_2 \otimes a_2) = \sum_{(a_1),(b_2)} b_1 b_2^{(2)} \otimes a_1^{(2)} a_2 \overline{\sigma}(a_1^{(1)}, b_2^{(1)}) \sigma(a_1^{(3)}, b_2^{(3)}) \qquad (2)$$

for $a_1, a_2 \in \mathcal{A}$ *and* $b_1, b_2 \in \mathcal{B}$. *With the tensor product coalgebra structure of* $\mathcal{B} \otimes \mathcal{A}$ *(with the coproduct* $\Delta_{\mathcal{B}\otimes\mathcal{A}}(b \otimes a) = \sum_{(a),(b)} b^{(1)} \otimes a^{(1)} \otimes b_{(2)} \otimes a^{(2)}$ *and counit* $\epsilon_{\mathcal{B}\otimes\mathcal{A}}(b \otimes a) = \epsilon_{\mathcal{B}}(b)\epsilon_{\mathcal{A}}(a)$*). This algebra is a bialgebra.*
ii) If $\mathcal{A}$ *and* $\mathcal{B}$ *are Hopf algebras, this bialgebra is a Hopf algebra with antipode* S *given by* $S(b \otimes a) = (1 \otimes S_{\mathcal{A}}(a))(S_{\mathcal{B}}(b) \otimes 1)$.

Proof.

$$\begin{aligned}
&((b_1 \otimes a_1)(b_2 \otimes a_2))(b_3 \otimes a_3)\\
&= (\sum_{(a_1),(b_2)} b_1 b_2^{(2)} \otimes a_1^{(2)} a_2 \overline{\sigma}(a_1^{(1)}, b_2^{(1)}) \sigma(a_1^{(3)}, b_2^{(3)}))(b_3 \otimes a_3)\\
&= \sum_{(a_1),(a_2),(b_2),(b_3)} b_1 b_2^{(2)} b_3^{(2)} \otimes (a_1^{(2)} a_2)^{(2)} a_3 \overline{\sigma}(a_1^{(1)}), b_2^{(1)}) \sigma(a_1^{(3)}, b_2^{(3)})\\
&\qquad \overline{\sigma}((a_1^{(2)} a_2)^{(1)}, b_3^{(1)}) \sigma((a_1^{(2)} a_2)^{(3)}, b_3^{(3)})\\
&= \sum_{(a_1),(a_2),(b_2),(b_3)} b_1 b_2^{(2)} b_3^{(2)} \otimes a_1^{(3)} a_2^{(2)} a_3 \overline{\sigma}(a_1^{(1)}), b_2^{(1)}) \sigma(a_1^{(5)}, b_2^{(3)}) \overline{\sigma}(a_1^{(2)} a_2^{(1)}, b_3^{(1)}) \sigma(a_1^{(4)} a_2^{(3)}, b_3^{(3)})\\
&= \sum_{(a_1),(a_2),(b_2),(b_3)} b_1 b_2^{(2)} b_3^{(2)} \otimes a_1^{(3)} a_2^{(2)} a_3 \overline{\sigma}(a_1^{(1)}), b_2^{(1)}) \sigma(a_1^{(5)}, b_2^{(3)}) \overline{\sigma}(a_1^{(2)}, (b_3^{(1)})^{(2)}) \overline{\sigma}(a_2^{(1)}, (b_3^{(1)})^{(1)})\\
&\qquad \sigma(a_1^{(4)}, (b_3^{(3)})^{(1)}) \sigma(a_2^{(3)}, (b_3^{(3)})^{(2)})\\
&= \sum_{(a_1),(a_2),(b_2),(b_3)} b_1 b_2^{(2)} b_3^{(3)} \otimes a_1^{(3)} a_2^{(2)} a_3 \overline{\sigma}(a_1^{(1)}, b_2^{(1)}) \overline{\sigma}(a_1^{(2)}, b_3^{(2)})\\
&\qquad \sigma(a_1^{(4)}, b_3^{(4)}) \sigma(a_1^{(5)}, b_2^{(3)}) \overline{\sigma}(a_2^{(1)}, b_3^{(1)}) \sigma(a_2^{(3)}, b_3^{(5)})
\end{aligned}$$

The other one calculates similar. This proves associativity. □

Definition 8.4. The bialgebra from the Prop. 8.3 is called the **generalized quantum double** of the bialgebras $\mathcal{A}$ und $\mathcal{B}$ with repsect to the skew-pairing σ or, briefly, the **quantum double** of $\mathcal{A}$ and $\mathcal{B}$. It is denoted by $\mathcal{D}(\mathcal{A}, \mathcal{B}, \sigma)$ or simply by $\mathcal{D}(\mathcal{A}, \mathcal{B})$ if no confusion can arise.

We give another isomorphic variant of the quantum double. $\mathcal{D}(\mathcal{B}, \mathcal{A}; \overline{\sigma}_{21})$ and $\mathcal{D}(\mathcal{A}, \mathcal{B}; \sigma)$ occur as generalized quantum doubles of the bialgebras $\mathcal{A}$ and $\mathcal{B}$. The bialgebras $\mathcal{D}(\mathcal{B}, \mathcal{A}; \overline{\sigma}_{21})$ and $\mathcal{D}(\mathcal{A}, \mathcal{B}; \sigma)$ are indeed isomorphic with isomorphism θ defined by

$$\theta(b \otimes a) = \sum_{(a),(b)} a^{(2)} \otimes b^{(2)} \sigma(a^{(1)}, b^{(1)}) \overline{\sigma}(a^{(3)}, b^{(3)}), \ a \in \mathcal{A}, b \in \mathcal{B}. \qquad (3)$$

Remark 8.5 (Dual algebra). Let $f, f_1, f_2 \in \mathcal{A}^*$ and $a, a_1, a_2 \in \mathcal{A}^*$. Recall that the dual vector space $\mathcal{A}^*$ to a finite dimensional algebra $\mathcal{A}$ is an algebra with respect to the multiplication $f_1 f_2(a) := (f_1 \otimes f_2)\Delta(a) = \sum_{(a)} f_1(a^{(1)}) f_2(a^{(2)})$. For $f \in \mathcal{A}^*$ define a functional $\Delta(f) \in (\mathcal{A} \otimes \mathcal{A})^*$ by $\Delta(f)(a_1 \otimes a_2) := (f \otimes M)(a_1 \otimes a_2) = f(a_1 a_2)$. $\mathcal{A}^*$ equipped with comultiplication Δ becomes a Hopf algebra. The antipode, the counit and the unit element of this Hopf algebra $\mathcal{A}^*$ are given by $(Sf)(a) = (f(S(a))), \epsilon_{\mathcal{A}^*}(f) = f(1)$ and $1_{A^*}(a) = \epsilon(a)$, respectively.

Now let $\mathcal{A}$ is a Hopf algebra with invertible antipode and let $\mu(f, a)$ denote the evaluation $f(a)$ of the functional $f \in \mathcal{A}^*$ at $a \in \mathcal{A}$. In the remark we already noted, that μ fulfills the property of a dual pairing. Since the antipode of $\mathcal{A}$ is invertible, A^{op} and $\mathcal{A}^{cop}$ are Hopf algebras and so are

$$(\mathcal{A}^{op})^* \equiv (\mathcal{A}^*)^{cop} \text{ and } (\mathcal{A}^{cop})^* \equiv (\mathcal{A}^*)^{op}.$$

Hence μ is a skew-pairing of $(\mathcal{A}^*)^{cop}$ and A and by (1) its convolution inverse $\overline{\mu}$ is given by $\overline{\mu}(f, a) = \mu(f, S^{-1}(a)9$. Therefore by 8.3 the quantum doubles $\mathcal{D}((\mathcal{A}^*)^{cop}, \mathcal{A}; \mu)$ and $\mathcal{D}(\mathcal{A}, (\mathcal{A}^*)^{cop}; \overline{\mu}_{21})$ are well defined Hopf algebras. Further, since μ is a dual pairing of $\mathcal{A}^*$ and $\mathcal{A}$, $\nu := \mu_{21}$ is a skew -pairing of $\mathcal{A}$ and $(\mathcal{A}^*)^{op}$ with convolution inverse, such that $\overline{\nu}(a, f) = \langle f, S(a) \rangle$. Thus the corresponding quantum doubles $\mathcal{D}(\mathcal{A}, (A^*)^{op}; \nu)$ and $\mathcal{D}((A^*)^{op}, \mathcal{A}; \overline{\nu}_{21})$ are well-defined. Each of the four Hopf algebras

$$\mathcal{D}((\mathcal{A}^*)^{cop}, \mathcal{A}; \mu), \mathcal{D}(\mathcal{A}, (\mathcal{A}^*)^{cop}; \overline{\mu}_{21}),$$

$$\mathcal{D}(\mathcal{A}, (A^*)^{op}; \nu) \text{ and } \mathcal{D}((A^*)^{op}, \mathcal{A}; \overline{\nu}_{21})$$

is called a **quantum double of the Hopf algebra** $\mathcal{A}$. All four Hopf algebras are in fact isomorphic with θ given in (3) and $S_{\mathcal{A}^\circ} \otimes id$.
Suppose that $\mathcal{A}$ is a finite-dimensional Hopf algebra. Let
$\{e_i \mid i = 1, 2, ..., n\}$ and $\{f_i \mid i = 1, 2, ..., n\}$ be bases of the vector spaces $\mathcal{A}$ and $\mathcal{A}^* \equiv \mathcal{A}^*$ respectively, such that $\langle f_j, e_i \rangle = \delta_{ij}$.

Theorem 8.6. *The quantum double $\mathcal{D}(\mathcal{A}) := \mathcal{D}(\mathcal{A}, (\mathcal{A}^*)^{cop}; \overline{\mu}_{21})$ is a quasitriangular Hopf algebra with the universal R-matrix*

$$\mathcal{R} = \sum_i (\epsilon \otimes e_i) \otimes (f_i \otimes 1) \in \mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{A}) \tag{4}$$

Corollary 8.7. *Any finite-dimensional Hopf algebra $\mathcal{A}$ is isomorphic to a Hopf subalgebra of a quasitriangular Hopf algebra.*

Proof. $\mathcal{A}$ is isomorphic to the Hopf subalgebra $1 \otimes \mathcal{A}$ of $\mathcal{D}(\mathcal{A})$. $\square$

Example 8.8. We want to apply the quantum double construction to the finite dimensional cocommutative Hopf algebra $k[G]$, where G is a finite group.

Let $\{e_g\}_{g\in G}$ be the dual basis of the basis $\{g\}_{g\in G}$ of $k\left[G\right]$. It is easy to check, that the dual algebra is the algebra k^G with multiplication given by

$$e_g e_h = \delta_{gh} e_g$$

for all $g, h \in G$ and with unit $\sum_{g\in G} e_g = 1$. The comultiplication Δ, the counit ϵ, and the antipode S of $(k\left[G\right]^{op})^*$ are defined by

$$\Delta(e_g) = \sum_{uv=g} e_v \otimes e_u,\ \ \epsilon(e_g) = \delta_{g1}\ S(e_g) = e_{g^{-1}}$$

for each element g of the group.
The definition of the quantum double shows that the set $\{e_g h\}_{(g,h)\in G\times G}$ is a basis of $\mathcal{D}(k\left[G\right], (k\left[G\right]^{op})^*)$. The product of the quantum double is determined by

$$he_g = e_{hgh^{-1}}h.$$

Its universal R-matrix is given by

$$\mathcal{R} = \sum_{g\in G}(\epsilon \otimes g) \otimes (e_g \otimes 1).$$

Despite the fact that the quantum double is not cocommutative when G is not abelian, its antipode is involutive.