R-matrices and Yang-Baxter equation

Recall: If (C, Δ, ε) is a coalgebra, we defined the opposite coproduct $\Delta^{\mathrm{op}} = \tau_{C,C} \circ \Delta$.

Definition. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We call it quasi-cocommutative if there exists an (multiplicative) invertible element $R \in H \otimes H$ such that for all $x \in H$ the equation

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \tag{$\star$}$$

holds. R is called universal R-matrix.
We define a quasi-cocommutative Hopf algebra as a Hopf algebra whose underlying bialgebra has a universal R-matrix.

Remark.

- In general, the universal R-matrix of a quasi-cocommutative bialgebra is not unique.
- Any cocommutative bialgebra is a quasi-cocommutative bialgebra with the universal R-matrix $R = 1 \otimes 1$.
- In Sweedler's sigma notation, the defining property of a universal R-matrix $R = \sum_i s_i \otimes t_i$ reads

$$\sum_{(x),i} x'' s_i \otimes x' t_i = \sum_{(x),i} s_i x' \otimes t_i x''$$

Notation. Let H be an algebra and $R = \sum_i s_i \otimes t_i$. Then we define three elements of $H \otimes H \otimes H$, namely

$$R_{31} = \sum_i t_i \otimes 1 \otimes s_i$$
$$R_{13} = \sum_i s_i \otimes 1 \otimes t_i$$
$$R_{12} = \sum_i s_i \otimes t_i \otimes 1$$

Definition. A quasi-cocommutative bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called braided (or quasi-triangular) if the universal R-matrix satisfies both

$$(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}$$
$$(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12}$$

Example.

- Any cocommutative bialgebra is braided with $R = 1 \otimes 1$ since $R_{ij} = 1 \otimes 1 \otimes 1$ for all $i, j \in \{1, 2, 3\}$ and $\Delta(1) = 1 \otimes 1$.
- *Sweedler's 4-dimensional Hopf algebra*:
  Let H be the algebra generated by x, y with the following relations:
  $$x^2 = 1, \quad y^2 = 0, \quad yx + xy = 0$$
  Then H is a vector space with basis $\{1, x, y, xy\}$ and uniquely becomes a Hopf algebra via
  $$\begin{array}{rll} \Delta(x) = x \otimes x & \varepsilon(x) = 1 & S(x) = x \\ \Delta(y) = 1 \otimes y + y \otimes x & \varepsilon(y) = 0 & S(y) = xy \end{array}$$
  It is braided with
  $$R_\lambda = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \frac{\lambda}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y)$$
  for any $\lambda \in K$. Furthermore, note that $S^2(a) = xax^{-1}$ and $(R_\lambda)^{-1} = \tau_{H,H}(R_\lambda)$.

Definition. Let V be a vector space over a field k. An R-matrix is a linear automorphism $c \in \mathrm{Aut}(V \otimes V)$ which solves the Yang-Baxter equation
$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)$$
that holds in $\mathrm{Aut}(V \otimes V \otimes V)$.

In terms of a given basis $\{v_i\}$ of V, we have
$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l$$
and therefore
$$(c \otimes \mathrm{id}_V)(v_i \otimes v_j \otimes v_k) = \sum_{p,q,r} c_{ij}^{pq} \delta_k^r v_p \otimes v_q \otimes v_r$$
So in terms of a basis, for all i, j, k the Yang-Baxter equation reads
$$\sum_{p,q,r,x,y,z,l,m,n} \left(c_{ij}^{pq}\delta_k^r\right)\left(\delta_p^x c_{qr}^{yz}\right)\left(c_{xy}^{lm}\delta_z^n\right) v_l \otimes v_m \otimes v_n = \sum_{p,q,r,x,y,z,l,m,n} \left(\delta_i^p c_{jk}^{qr}\right)\left(c_{pq}^{xy}\delta_r^z\right)\left(\delta_x^l c_{yz}^{mn}\right) v_l \otimes v_m \otimes v_n$$
Since $\{v_l \otimes v_m \otimes v_n\}_{l,m,n}$ forms a basis of $V \otimes V \otimes V$, both terms must equal for all l, m, n. So we have for all i, j, k, l, m, n:
$$\sum_{p,q,r,x,y,z} \left(c_{ij}^{pq}\delta_k^r\right)\left(\delta_p^x c_{qr}^{yz}\right)\left(c_{xy}^{lm}\delta_z^n\right) = \sum_{p,q,r,x,y,z} \left(\delta_i^p c_{jk}^{qr}\right)\left(c_{pq}^{xy}\delta_r^z\right)\left(\delta_x^l c_{yz}^{mn}\right)$$
$$\Leftrightarrow \sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{q,r,y} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn}$$

Theorem 1. *Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra.*

1. *The universal R-matrix R satisfies*
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1}$$
and the equation
$$(\varepsilon \otimes id_H)(R) = 1 = (id_H \otimes \varepsilon)(R)$$
holds.

2. *If H is a Hopf algebra whose antipode is invertible, then*
$$(S \otimes id_H)(R) = R^{-1} = \left(id_H \otimes S^{-1}\right)(R)$$
and
$$(S \otimes S)(R) = R.$$

Equation (1) is equivalent to

$$\sum_{i,j,k} s_k s_j \otimes t_k s_i \otimes t_j t_i = \sum_{i,j,k} s_j s_i \otimes s_k t_i \otimes t_k t_j$$

Next we want to show that the universal R-matrix of a braided bialgebra gives a solution to the Yang-Baxter equation. Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ be a braided bialgebra and let V, W be two H-modules. Then we define the map $c^R_{V,W} \colon V \otimes W \to W \otimes V$ by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}\left(R\left(v \otimes w\right)\right).$$

If we denote the universal R-matrix as $R = \sum_i s_i \otimes t_i$, this equation reads

$$c^R_{V,W}\left(v \otimes w\right) = \sum_i t_i w \otimes s_i v.$$

Proposition 2. *Let $(H, \mu, \eta, \Delta, \varepsilon, R)$ and $c^R_{V,W}$ be as above.*
Then

1. *$c^R_{V,W}$ is an isomorphism of H-modules and*

2. *for any three H-modules U, V, W the identities*
$$\begin{aligned} c^R_{U\otimes V,W} &= \left(c^R_{U,W} \otimes \mathrm{id}_V\right)\left(\mathrm{id}_U \otimes c^R_{V,W}\right) \\ c^R_{U,V\otimes W} &= \left(\mathrm{id}_V \otimes c^R_{U,W}\right)\left(c^R_{U,V} \otimes \mathrm{id}_W\right) \end{aligned}$$
$$\left(c^R_{V,W} \otimes \mathrm{id}_U\right)\left(\mathrm{id}_V \otimes c^R_{U,W}\right)\left(c^R_{U,V} \otimes \mathrm{id}_W\right) = \left(\mathrm{id}_W \otimes c^R_{U,V}\right)\left(c^R_{U,W} \otimes \mathrm{id}_V\right)\left(\mathrm{id}_U \otimes c^R_{V,W}\right)$$
hold.

If we now set $U = V = W$, $c^R_{V,V}$ is a solution of the Yang-Baxter equation, since

$$\left(c^R_{V,V} \otimes \mathrm{id}_V\right)\left(\mathrm{id}_V \otimes c^R_{V,V}\right)\left(c^R_{V,V} \otimes \mathrm{id}_V\right) = \left(\mathrm{id}_V \otimes c^R_{V,V}\right)\left(c^R_{V,V} \otimes \mathrm{id}_V\right)\left(\mathrm{id}_V \otimes c^R_{V,V}\right).$$