6 Seminar on Hopf algebras: Modules, tensor products and duals

Definition 6.1. Let R be a ring. A left R-module is an abelian group M together with a bilinear map

$$\cdot : R \times M \to M,\ (r, m) \mapsto r.m$$

such that for all $r, s \in R,\ m, n \in M$

$$(rs).m = r.(s.m) \text{ and } 1.m = m$$

A R-module homomorphism between left R-modules M and N is a group homomorphism $f : M \to N$ such that for all $r \in R,\ m \in M : f(r.m) = r.f(m)$

Definition 6.2. Let k be a field and A a k-algebra. A representation of A on a k-vectorspace V is an algebra homomorphism

$$\rho : A \to \mathrm{End}_k(V).$$

An intertwiner between two representations ρ, σ of A over k-vectorspaces V and W, respectively, is a k-linear map $\alpha : V \to W$, such that

$$\alpha \circ \rho(a) = \sigma(a) \circ \alpha$$

for all $a \in A$.

Definition 6.3. Let A be a k-algebra and V an abelian group. We say V is a left A-module, if it possesses the induced vectorspace structure $\lambda.v := \eta_A(\lambda).v$ and a k-linear map $\cdot : A \otimes M \to M$, that yields the previous defined A-module structure, with A seen as a ring.

Remark 6.4. Let k be a field and A a k-algebra. We have a bijection between $\{(V, \rho) \mid \rho$ representation of A on k-vectorspace $V\}$ and $\{(V, \cdot) \mid V$ left A-module$\}$ by $a.v := \rho(a)(v)$ and $\rho(a)(v) := a.v$.

Remark 6.5. Let A be an algebra. If U, V are A-modules, the vector space $U \otimes V$ obtains an $A \otimes A$-module structure by

$$(a \otimes a').(u \otimes v) := a.u \otimes a'.v.$$

If A is a bialgebra, the $A \otimes A$-module $U \otimes V$ is also an A-module by

$$a.(u \otimes v) := \Delta(a).(u \otimes v) = \sum_{(a)} (a'.u \otimes a''.v).$$

Example 6.6. Let $(G, \cdot)$ be a group. Let $h \in G$. We can make k into a kG-module k_h by defining on bases

$$\delta_g.1 = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

For $g, h \in G$, we have a kG-module isomomorphism

$$\varphi : k_g \otimes k_h \to k_{gh}$$
$$\lambda \otimes \mu \mapsto \lambda\mu$$

where the tensor product on the left is given the induced kG-module structure previously defined.

PROPOSITION 6.7. *If A is a bialgebra, U, V and W are A-modules and k is given the trivial A-module structure, we have canonical A-module isomorphisms*

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

and

$$k \otimes V \cong V \cong V \otimes k.$$

If furthermore A is cocommutative, then $V \otimes W \overset{\tau_{V,W}}{\cong} W \otimes V$ is also an A-module isomorphism.

REMARK 6.8. Let A be a Hopf algebra with antipode $S \in \mathrm{End}(A)$ and let V, W be A-modules. We can give $\mathrm{Hom}_k(V, W)$ an A-module structure: first, it obtains an $A \otimes A^{\mathrm{op}}$-module structure by

$$((a \otimes a').f)(v) := a.f(a'.v).$$

Then $(id_{\mathrm{A}} \times S) \circ \Delta : A \to A \otimes A^{\mathrm{op}}$ is an algebra morphism. Define the action of A on $\mathrm{Hom}_k(V, W)$ by

$$(a.f)(v) := ((((id_{\mathrm{A}} \times S) \circ \Delta)(a)).f)(v) = \sum_{(a)} a'.f(S(a'').v).$$

PROPOSITION 6.9. *Let A be a Hopf algebra and U, U', V, V' be A-modules. Then the linear map*

$$\lambda : \mathrm{Hom}_k(U, U') \otimes \mathrm{Hom}_k(V, V') \to \mathrm{Hom}_k(V \otimes U, U' \otimes V')$$

defined by

$$(\lambda(f \otimes g))(v \otimes u) = f(u) \otimes g(v)$$

is A-linear if, in addition, the flip $\tau_{U^*,V'} : U^* \otimes V' \to V' \otimes U^*$ is A-linear.

COROLLARY 6.10. *In particular, the map $\lambda : U^* \otimes V^* \to (V \otimes U)^*$ and $\lambda_{U,V} : V \otimes U^* \to \mathrm{Hom}_k(U, V)$ with $(\lambda_{U,V}(v \otimes u'))(u) = u'(u).v$ are A-linear.*

REMARK 6.11. If either U or U', and, either V or V', are finite-dimensional vector spaces, λ is invertible. In particular, the map $\lambda_{U,V}$ is invertible, if U or V are finite-dimensional.

PROPOSITION 6.12. *Let V be an A-module. Then the evaluation map $ev_V : V^* \otimes V \to k, f \otimes v \mapsto f(v)$ is A-linear. If, moreover, the vector space V is finite-dimensional, then the k-linear coevaluation map $\delta_V : k \to V \otimes V^*$, $1 \mapsto (\lambda_{V,V}^{-1} \circ \eta_{\mathrm{End}_k(V)})(1)$ and the composition*

$$\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(U, V) \overset{\circ}{\to} \mathrm{Hom}(U, W)$$

are A-linear too.