Hopf algebras

Lemma 1. *Let H be a bialgebra and $S : H \to H^{op}$ be an algebra morphism. Assume that H is generated as an algebra by a subset X such that,*

$$\sum_{(x)} x' S(x'') = \epsilon(x)1 = \sum_{(x)} S(x')x''$$

for all $x \in X$. Then S is an antipode for H.

Example 1. *We already know the tensor algebra $H = T(V)$ for a vector space V. This bialgebra becomes a Hopf algebra via $S : H \to H^{op}$ with $S(1) = 1$ and for all $v_1, v_2, ..., v_n \in V$*

$$S(v_1 v_2 ... v_n) = (-1)^n v_n ... v_2 v_1.$$

As in the cases of algebras, coalgebras and bialgebras we too have a notion of ideals in Hopf algebras.

Definition 1. *Let H be a Hopf algebra and $I \subset H$ a bi-ideal, i.e. a ideal and coideal in H. If we also have that $S(I) \subset I$ we call I a* Hopf-ideal.

We have the following Theorem which is similar to the previously mentioned cases.

Theorem 1. *Let H, F be Hopf algebras, $f : F \to H$ a Hopf algebra map and $I \subset F$ a Hopf-ideal. Also let $\pi : F \to F/I$ be the natural projection. Then the following hold:*

a) F/I has a unique Hopf algebra structure such that π is a Hopf algebra map.

b) $\ker(f)$ is a Hopf-ideal.

c) If we have $I \subset \ker(f)$ then there is a unique Hopf algebra map

$$\bar{f} : F/I \to H$$

with $\bar{f} \circ \pi = f$.

Let k be a field. As a reminder let us look at the group algebra $k[G]$ for some group G.

Example 2. *The algebra $k[G]$ has a coalgebra structure, which is defined by*

$$\Delta(g) = g \otimes g \quad and \quad \epsilon(g) = 1 \forall g \in G$$

Question: Are there elements with those properties in other Hopf algebras? Do they have special properties?

Definition 2. *An element h of a bialgebra H is called* grouplike *if it satisfies $\Delta(h) = h \otimes h$. We denote by $\mathcal{G}(H)$ the set of grouplike elements.*

Note 1. *It follows that $\epsilon(g) = 1 \quad \forall g \in \mathcal{G}(H)$*

Proposition 1. *Let H be a bialgebra. Then $\mathcal{G}(H)$ is a monoid with 1 for the multiplication of H. If H is a Hopf algebra, i.e. if H has a invertible antipode S, $\mathcal{G}(H)$ even becomes a group.*

Let us look at $k[G]$ again.

Example 3. *As one would expect we have $\mathcal{G}(H) = G$ in the Hopf algebra $H = k[G]$.*

Definition 3. *Let H be a Hopf algebra and $g, h \in \mathcal{G}(H)$ grouplike. We say an element $x \in H$ is* (g, h)-primitive *if we have $\Delta(x) = x \otimes g + h \otimes x$. A $(1, 1)$-primitive element is just called* primitive element.

Example 4. *Two examples of primitive elements in algebras we know:*

- *The basis Element x in last times example of Sweedler's 4-dimensional Hopf algebra is $(c, 1)$-primitive.*
- *The generators of the Tensoralgebra are primitive.*

Proposition 2. *Let H be a Hopf algebra and $x, y \in H$ primitive. We have $\epsilon(x) = 0$ and the commutator $[x, y] = xy - yx$ is also primitive.*

Proposition 3. *Let H be a finite dimensional Hopf algebra over a field k of characteristic zero. If $x \in H$ is a primitive Element, then $x = 0$.*

Proposition 4. *Let V be a vector space with basis $\{v_1, v_2, ..., v_n\}$ and H a Hopf algebra. If $x_1, x_2, ..., x_n \in H$ are primitive Element, we have a unique Hopf algebra morphism $f : T(V) \to H$, such that $f(v_i) = x_i$.*

Theorem 2. *Let V be a vectorspace and $T(V)$ the tensor algebra of V. For generators of $v_i \in i(V) \subset T(V)$, $1 \leq i \leq n$, for $n \in \mathbb{N}$, of $T(V)$ we have:*

$$\epsilon(v_1...v_n) = 0$$

and

$$\Delta(v_1...v_n) = 1 \otimes v_1...v_n + \sum_{p=1}^{n-1} \sum_{\sigma} (v_{\sigma(1)}...v_{\sigma(p)} \otimes v_{\sigma(p+1)}...v_{\sigma(n)}) + v_1...v_n \otimes 1,$$

where $\sigma \in S_n$ runs over all permutations such that:

$$\sigma(1) < \sigma(2) < ... < \sigma(p)$$

and

$$\sigma(p+1) < \sigma(p+2) < ... < \sigma(n)$$

such a σ is called a $(p, n-p)-$shuffle.

We now see that by Prop.4 for any set $x_1, ..., x_n$ of primitive elements, $\Delta(x_1...x_n)$ is given by the formula in the previous theorem replacing v_i with x_i.