4 Hopf Algebras

4.1 Hopf Algebras

Let (C, Δ, ϵ) a coalgebra and (A, M, u) an algebra. We define on the set $\mathrm{Hom}(C, A)$ an algebra structure in which the multiplication

$$* : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A), \quad f \otimes g \mapsto M \circ (f \otimes g) \circ \Delta$$

is given as follows: for $f, g \in \mathrm{Hom}(C, A)$

$$(f * g)(c) = \sum_{(c)} f(c^{(1)})g(c^{(2)})$$

for any $c \in C$. This multiplication is associative, since for $f, g, h \in \mathrm{Hom}(C, A)$ and $c \in C$ we have

$$\begin{aligned} ((f * g) * h)(c) &= \sum_{(c)} (f * g)(c^{(1)})h(c^{(2)}) = \sum_{(c)} f(c^{(1)})g(c^{(2)})h(c^{(3)}) \\ &= \sum_{(c)} f(c^{(1)})(g * h)(c^{(2)}) = (f * (g * h))(c) \end{aligned}$$

The identity element of the algebra $\mathrm{Hom}(C, A)$ is $u\epsilon \in \mathrm{Hom}(C, A)$, since

$$(f * (u\epsilon))(c) = \sum_{(c)} f(c^{(1)})(u\epsilon)(c^{(2)}) = \sum_{(c)} f(c^{(1)})\epsilon(c^{(2)})u(1) = \sum_{(c)} f(c^{(1)})\epsilon(c^{(2)})1 = f(c)$$

hence $f * (u\epsilon) = f$ and similarly $(u\epsilon) * f = f$.

If we consider the dual algebra C^* of a coalgebra C, we have the multiplication $M : C^* \otimes C^* \to C^*$ on C^* given by $M = \Delta^* \circ \rho$. If we denote $M(f \otimes g)$ by $f * g$ we obtain

$$(f * g)(c) = (\Delta^* \rho)(f \otimes g)(c) = \rho(f \otimes g)(\Delta(c)) = \sum_{(c)} f(c^{(1)})g(c^{(2)})$$

for $f, g \in C^*$ and $c \in C$. We call this multiplication convolution product.
If $A = k$, then the product $*$ on the algebra $\mathrm{Hom}(C, k)$ is the same as the convolution product defined on the dual algebra C^* of the coalgebra C. This is why in the case A is an arbitrary algebra we will also call $*$ the <u>convolution product</u>.

In the following is H an bialgebra. We denote by H^c the underlying coalgebra, and by H^a the underlying algebra of H. Define as above an algebra structure on $\mathrm{Hom}(H^c, H^a)$, in which the multiplication is defined as the convolution product. Remark that the identity $\mathrm{I} : H \to H$ is an element of $\mathrm{Hom}(H^c, H^a)$.

Definition 4.1 *Let H be a bialgebra. A linear map $S : H \to H$ is called an antipode of the bialgebra H if S is the inverse of the identity map I with respect to the convolution product in* $\mathrm{Hom}(H^c, H^a)$.

Definition 4.2 *A bialgebra H having an antipode is called a <u>Hopf algebra</u>.*

Remark 4.3. *In a Hopf algebra the antipode is unique, being the inverse of the element I in the algebra $\mathrm{Hom}(H^c, H^a)$. The fact that $S : H \to H$ is the antipode can be written as $S * I = I * S = u\epsilon$ and using the sigma notation*

$$\sum_{(h)} S(h^{(1)})h^{(2)} = \sum_{(h)} h^{(1)}S(h^{(2)}) = \epsilon(h)u(1)$$

for any $h \in H$.

Since H is a bialgebra, we keep the convention to say that a Hopf algebra has a property P if the underlying algebra or coalgebra has the property P.

Definition 4.4 *Let H and B be two Hopf algebras. A map $f : H \to B$ is called a morphism of Hopf algebras if it is a morphism of bialgebras.*

Proposition 4.5. *Let H and B be two Hopf algebras with antipodes S_H and S_B. If $f : H \to B$ is a bialgebra map, then $S_B f = f S_H$.*

Proposition 4.6. *Let H be a Hopf algebra with antipode S. Then:*

1. *$S(hg) = S(g)s(h)$ for any $g, h \in H$.*
2. *$S(1) = 1$.*
3. *$\Delta(S(h)) = \sum_{(h)} S(h^{(2)}) \otimes S(h^{(1)})$.*
4. *$\epsilon(S(h)) = \epsilon(h)$.*

Which means that the antipode of a Hopf algebra H is an antimorphism of algebras and coalgebras.

Proposition 4.7. *Let H be a Hopf algebra with antipode S. Then the following assertions are equivalent:*

1. *$\sum_{(h)} S(h^{(2)})h^{(1)} = \epsilon(h)1$ for any $h \in H$.*
2. *$\sum_{(h)} h^{(2)}S(h^{(1)}) = \epsilon(h)1$ for any $h \in H$.*
3. *$S^2 = I$ ($S^2 := S \circ S$).*

Corollary 4.8. *let H be a commutative or cocommutative Hopf algebra. Then $S^2 = I$.*

We have already seen that if H is a finite dimensional bialgebra, then its dual is a bialgebra. The following result shows that if H is even a Hopf algebra, then its dual also has a Hopf algebra structure.

Proposition 4.9. *Let H be a finite dimensional Hopf algebra, with antipode S. Then the bialgebra H^* is a Hopf algebra, with antipode S^*.*

4.2 Examples

Example 4.10. *If H and L are two bialgebras, then it is easy to check that we have a bialgebra structure on $H \otimes L$ if we consider the tensor product of algebras and the tensor product of coalgebras structures. Moreover, if H and L are Hopf algebras with antipodes S_H and S_L, then $H \otimes L$ is a Hopf algebra with antipode $S_H \otimes S_L$. This bialgebra (Hopf algebra) is called the tensor product of the two bialgebras (Hopf algebras).*

Example 4.11 (The group algebra)**.** *Let G be a multiplicative group, and $k[G] := \bigoplus_{g\in G} kg$ group algebra. This is a k-vector space with basis $\{b_g | b_g := g \in G\}$, so its elements are of the form $\sum_{g\in G} \alpha_g b_g$ with $(\alpha_g)_{g\in G} \subset k$ with only a finite number of non-zero elements. The multiplication is defined on the basis by*

$$b_g \cdot b_h = b_{g\cdot h}$$

for $g, h \in G$. On the group algebra $k[G]$ we also have a coalgebra structure, by $\Delta(b_g) = b_g \otimes b_g$, and $\epsilon(b_g) = 1$ for any $g \in G$. We already know that the group algebra becomes in this way a bialgebra. We note that until now we only used the fact that G is a monoid. The existence of the antipode is directly related to the fact that the elements of G are invertible. Indeed, the map $S : k[G] \to k[G]$ defined by $S(b_g) = b_{g^{-1}}$, and then extended linearly, is an antipode for the bialgebra $k[G]$, since

$$\sum_{(b_g)} S(b_g^{(1)}) b_g^{(2)} = S(b_g) b_g = b_{g^{-1}} b_g = 1 = \epsilon(b_g)1$$

and similarly, $\sum_{b_g} b_g^{(1)} S(b_g^{(2)}) = \epsilon(b_g)1$ for any $g \in G$. It is clear that if G is a monoid, which is not a group, then the bialgebra $k[G]$ is not a Hopf algebra.
If G is a finite group, then we know by Proposition 4.9 that on $(k[G])^*$ we also have a Hopf algebra structure, which is dual to the one on $k[G]$. We recall that the algebra $(k[G])^*$ has a basis, that is the dual basis to the basis on $k[G]$, $(p_g)_{g\in G}$, where $p_g \in (k[G])^*$ is defined by $p_g(h) = \delta_{b_g, b_h}$ for any $g, h \in G$. Therefore,

$$p_g^2 = p_g, \; p_g p_h = 0 \; \text{ for any } g \neq h, \; \sum_{g\in G} p_g = 1_{(k[G])^*}.$$

The coalgebra structure of $(k[G])^*$ is given by

$$\Delta(p_g) = \sum_{x\in G} p_x \otimes p_{x^{-1}g}, \quad \epsilon(p_g) = \delta_{1,g}.$$

The antipode of $(k[G])^*$ is defined by $S^*(p_g) = p_{g^{-1}}$ for any $g \in G$.

Example 4.12 (Sweedler's 4-dimensional Hopf algebra)**.** *Assume that* $\mathrm{char}(k) \neq 2$. *Let H be the algebra given by the generators and relations as follows: H is created as a k-algebra by c and x satisfying the relations*

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx.$$

Then H has dimension 4 as a k-vector space, with basis $\{1, c, x, cx\}$. The coalgebra structure is induced by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1, \quad \epsilon(c) = 1, \quad \epsilon(x) = 0.$$

In this way H becomes a bialgebra, which also has an antipode S given by $S(c) = c^{-1}$ and $S(x) = -cx$.
This was the first example of a non-commutative and non-cocommutative Hopf algebra.