Bialgebras

1. Sweedler’s sigma notation

Let \((C, \Delta, \epsilon)\) be a coalgebra and \(x \in C\).

Then the element \(\Delta(x) \in C \otimes C\) is of the form \(\Delta(x) = \sum_i x'_i \otimes x''_i\). By omission of the subscript we write instead \(\Delta(x) = \sum (x) x'_1 \otimes x''_1\).

Using this notation we can rewrite the condition for coassociativity:
\[
\sum (x) \left( \sum (x') (x')' \otimes (x')'' \otimes x'' = \sum (x) x' \otimes \left( \sum (x'') (x'')' \otimes (x'')'' \right) \right).
\]
By convention we write for both sides of the above equation \(\sum (x) x'_1 \otimes x''_1 \otimes x'''_1\) or \(\sum (x) x^{(1)} \otimes x^{(2)} \otimes x^{(3)}\).

Applying the comultiplication to one of the components of the sum we get three equal expressions:
\[
\sum (x) \Delta(x') \otimes x''_1 \otimes x'''_1, \sum (x) x' \otimes \Delta(x'') \otimes x'''_1, \sum (x) x' \otimes x'' \otimes \Delta(x'''_1).
\]
For these we write \(\sum (x) x'_1 \otimes x''_1 \otimes x'''_1 \otimes x^{(4)}\) or \(\sum (x) x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}\).

More generally we inductively define maps \(\Delta^{(n)} : C \to C^{\otimes (n+1)}\) for \(n \geq 1\) by \(\Delta^{(1)} = \Delta\) and \(\Delta^{(n)} = (\Delta \otimes id_{\otimes (n-1)}) \circ \Delta^{(n-1)}\).

By convention we write \(\Delta^{(n)}(x) = \sum (x) x^{(1)} \otimes ... \otimes x^{(n+1)}\).

Using these conventions we can reformulate the condition for counitality as
\[
\sum (x) \epsilon(x') x'' = \sum (x) x' \epsilon(x'') \quad \text{for all } x \in C
\]
We get identities such as
\[
\sum (x) x^{(1)} \otimes \epsilon(x^{(2)}) \otimes x^{(3)} \otimes x^{(4)} \otimes x^{(5)} = \sum (x) x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)},
\]
by applying the reformulation of the counitality condition to the left-hand side rewritten as
\[
\sum (x) x^{(1)} \otimes (\epsilon \otimes id)(\Delta x^{(2)}) \otimes x^{(3)} \otimes x^{(4)}.
\]
We may further express the cocommutativity of the coalgebra \(C\) by
\[
\sum (x) x' \otimes x'' = \sum (x) x'' \otimes x' \quad \text{for all } x \in C.
\]
Also the relation \((f \otimes f) \circ \Delta = \Delta' \circ f\) for defining coalgebra morphisms can be reformulated as
\[
\sum (x) f(x') \otimes f(x'') = \sum (f(x)) f(x') \otimes f(x'').
\]
2. Bialgebras

Let $H$ be a vector space such that $(H, \mu, \eta)$ is an algebra and $(H, \Delta, \epsilon)$ is a coalgebra. We have

Theorem 1: The following statements are equivalent.

a) $\mu$ and $\eta$ are coalgebra morphisms.

b) $\Delta$ and $\epsilon$ are algebra morphisms.

Proof: We write down the commutative diagrams expressing that $\mu$ and $\eta$ are coalgebra morphisms.

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\mu} & H \\
(id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) & \downarrow & \downarrow \Delta \\
(H \otimes H) \otimes (H \otimes H) & \xrightarrow{\mu \otimes \mu} & H \otimes H \\
(\sigma \otimes \id) & \downarrow & \downarrow \id \\
K & \xrightarrow{\epsilon} & K
\end{array}
$$

and $\eta$ are coalgebra morphisms.

$$
\begin{array}{ccc}
K & \xrightarrow{\eta} & H \\
\downarrow \id & & \downarrow \Delta \\
K \otimes K & \xrightarrow{\eta \otimes \eta} & H \otimes H
\end{array}
$$

Now it is easy to see that these are the same as the ones expressing that $\Delta$ and $\epsilon$ are algebra morphisms:

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\mu} & H \\
(id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) & \downarrow & \downarrow \Delta \\
(H \otimes H) \otimes (H \otimes H) & \xrightarrow{\mu \otimes \mu} & H \otimes H \\
(\sigma \otimes \id) & \downarrow & \downarrow \id \\
H \otimes H & \xrightarrow{\epsilon \otimes \epsilon} & K \otimes K \\
\downarrow \mu & & \downarrow \id \\
K & \xrightarrow{\eta \otimes \eta} & H \otimes H \\
\end{array}
$$

Definition 1: A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ such that $(H, \mu, \eta)$ is an algebra, $(H, \Delta, \epsilon)$ is a coalgebra and one of the equivalent conditions of Theorem 1 is true.

Using Sweedler’s sigma notation we can rewrite the condition $\Delta(xy) = \Delta(x)\Delta(y)$ as follows:

$$
\sum_{(xy)} (xy)' \otimes (xy)'' = \sum_{(x)(y)} x'y' \otimes x''y''
$$

We also get $\Delta(1) = 1 \otimes 1$, $\epsilon(xy) = \epsilon(x)\epsilon(y)$, $\epsilon(1) = 1$.

We now introduce the opposite coalgebra.
Let \((C, \Delta, \epsilon)\) be a coalgebra. Consider the function \(\Delta^{\text{op}} = \tau_{C,C} \circ \Delta\) where \(\tau_{C,C}\) denotes the flip \(\tau_{C,C} : C \otimes C \rightarrow C \otimes C : c_1 \otimes c_2 \mapsto c_2 \otimes c_1\).

Then \(C^{\text{cop}} := (C, \Delta^{\text{op}}, \epsilon)\) is a coalgebra which we call the opposite coalgebra.

Similarly, if \((A, \mu, \eta)\) is an Algebra then \((A, \mu^{\text{op}}, \eta)\) is an algebra which we call the opposite algebra and denote by \(A^{\text{op}}\). This gives us the following result:

Let \(H = (H, \mu, \eta, \Delta, \epsilon)\) be a bialgebra. Then \(H^{\text{op}} = (H, \mu^{\text{op}}, \eta, \Delta, \epsilon)\), \(H^{\text{cop}} = (H, \mu, \eta, \Delta^{\text{op}}, \epsilon)\) and \(H^{\text{op cop}} = (H, \mu^{\text{op}}, \eta, \Delta^{\text{op}}, \epsilon)\) are also bialgebras.

**Theorem 2:** The dual of a finite dimensional bialgebra is again a bialgebra.

**Proof:** We know that the dual of any coalgebra is a coalgebra and that of any finite dimensional algebra is an algebra. All we need to do is show that the conditions of Theorem 1 are true. □

**Examples:**

Let \((G, \ast)\) be a group, \(C = K[G] := \oplus_{g \in G} Kg\) be the vector space with basis \(G\).

The group multiplication and its neutral element naturally make \(C\) an algebra.

We define a coalgebra structure on \(C\) via \(\Delta(x) = x \otimes x, \epsilon(x) = 1\).

Then we have
\[
\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y) \quad \text{and} \quad \epsilon(xy) = 1 = \epsilon(x)\epsilon(y).
\]

This shows that \(\Delta\) and \(\epsilon\) are algebra morphisms which makes \(K[G]\) a bialgebra.

The dual algebra \(C^* = K[G]^*\) is the algebra of functions on \(G\) with values in \(K\).

In case \(G\) is finite the dual of the finite dimensional algebra \(K[G]\) has a coalgebra structure and therefore \(K[G]^*\) again is a bialgebra.

Comultiplication and counit are given by
\[
\Delta(f)(x \otimes y) = f(xy) \quad \text{and} \quad \epsilon(f) = f(1).
\]

**Theorem 3:** Let \(K\) be a field, \(n \geq 2\). There is no bialgebra structure on \(M_n(K)\) such that the underlying algebra structure is that of the matrix algebra.

**Proof:** Suppose we had a bialgebra structure on \(M_n(K)\), then the counit \(\epsilon : M_n(K) \rightarrow K\) is an algebra morphism. The kernel of \(\epsilon\) is a two - sided ideal of \(M_n(K)\), so it has to be either 0 or all of \(M_n(K)\). Since \(\epsilon(1) = 1\) we have \(\ker(\epsilon) = 0\) and obtain a contradiction to \(\dim(M_n(K)) > \dim(K)\). □

The tensor bialgebra
Let $M$ be a $K$-vector space. Consider the tensor algebra $(T(M), i)$. We can define a coalgebra structure on $T(M)$:

Let $\alpha, \beta$ be elements of $T(M)$. By convention we write $\alpha \bar{\otimes} \beta \in T(M) \otimes T(M)$.

Consider the linear function $f : M \to T(M) \otimes T(M) : m \mapsto m \bar{\otimes} 1 + 1 \bar{\otimes} m$.

By application of the universal property of the tensor algebra we get an algebra morphism $\Delta : T(M) \to T(M) \otimes T(M)$ such that $\Delta i = f$.

Again for $g : M \to TM \otimes TM \otimes TM : m \mapsto m \bar{\otimes} 1 \bar{\otimes} 1 + 1 \bar{\otimes} m \bar{\otimes} 1 + 1 \bar{\otimes} 1 \bar{\otimes} m$ the same property ensures the existence of an unique map $\bar{g}$ such that the following diagram commutes:

\[ M \xrightarrow{i} TM \xrightarrow{f} TM \otimes TM \otimes TM \]

Since we have

$(\Delta \otimes I)\Delta(m) = (\Delta \otimes I)(m \bar{\otimes} 1 + 1 \bar{\otimes} m) = m \bar{\otimes} 1 \bar{\otimes} 1 + 1 \bar{\otimes} m \bar{\otimes} 1 + 1 \bar{\otimes} 1 \bar{\otimes} m = g(m)$

and $(I \otimes \Delta)\Delta(m) = (I \otimes \Delta)(m \bar{\otimes} 1 + 1 \bar{\otimes} m) = m \bar{\otimes} 1 \bar{\otimes} 1 + 1 \bar{\otimes} m \bar{\otimes} 1 + 1 \bar{\otimes} 1 \bar{\otimes} m = g(m)$

We therefore have $(\Delta \otimes I)\Delta(m) = (I \otimes \Delta)\Delta(m)$ which proves that $\Delta$ is coassociative.

For a counit we apply the universal property to the function $0 : M \to K$ and receive an algebra morphism $\epsilon : T(M) \to K$ with $\epsilon(m) = 0 \ \forall m \in i(M)$.

The same universality argument as above shows that $\epsilon$ is a counit.

This makes $T(M)$ a bialgebra.