1 Algebras

Definition 1.1. *Let A be a ring. A **left module** over A is an abelian group M with an operation of A on M with the following properties for all $a, b \in A$ and $x, y \in M$:*

i) $(a + b)x = ax + bx$

ii) $a(x + y) = ax + ay$

iii) $(ab)x = a(bx)$

Definition 1.2. *Let A be a commutative ring and M an A-module equipped with an additional binary operation from $M \times M$ to M denoted by $\circ$. Then M is an **algebra over A** if $\circ$ satisfies the following conditions for all $x, y, z \in M$ and for all $a, b \in A$:*

i) $(x + y) \circ z = x \circ z + y \circ z$

ii) $x \circ (y + z) = x \circ y + x \circ z$

iii) $(ax) \circ (by) = (ab)(x \circ y)$

*i.e. if $\circ$ is bilinear. If we furthermore have $(x \circ y) \circ z = x \circ (y \circ z)$ we call M **associative** and if there is an element $U \in M$ such that $U \circ x = x = x \circ U$ we call U **unit** and M **unital**. If we have a field K instead of the ring A, we call M an **algebra over the field K**.*

From now on we shall only consider unital associative Algebras over fields.

Example 1.3. *i) The $n \times n$ matrices over a field K form a unital associative K-algebra because matrixmultiplication is distributive, associative and compatible with scalar multiplication. The unit is given by the identity matrix.*

*ii) The **group algebra of a group G over a field K** is the K-vector space with basis G i.e. it is composed of all finite sums of the type $\sum_{g \in G} \alpha_g g, g \in G, \alpha_g \in K$. The operations are defined as follows:*

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g) g$$

$$(\sum_{g \in G} \alpha_g g)(\sum_{g \in G} \beta_g g) = \sum_{h \in G} \left(\sum_{\substack{xy = h \\ x, y \in G}} (\alpha_x \beta_y) h \right).$$

The sum on the right side of the second formula is also finite. It remains to check that this is associative, distributive, compatible with scalar multiplication and that there is a unit.

iii) Let S be a set then the set $Fun(S, K)$ of functions from S to K form a unital associative K-algebra by pointwise addition and multiplication. The unit is given by the function mapping all of S to $1 \in K$.

iv) ***Sweedler's 4-dimensional Hopf algebra:***
Consider a field K with $char(K) \neq 2$. Let H be the algebra given by generators and relations as follows. H is generated as a K-algebra by c and x satisfying the relations

$$c^2 = 1\ ,\ x^2 = 0\ ,\ xc = -cx.$$

Then H has dimension 4 as a K-vector space, with basis $\{1, c, x, cx\}$. Additional Properties of this algebra will be discussed in later talks.

Definition 1.4. *A map $f : A \to B$ between two unital associative Algebras A, B over the same field K is called* ***algebra homomorphism****, if the following holds for all $k \in K$ and $x, y \in A$*

$$\begin{aligned} f(kx) &= kf(x) \\ f(x+y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \\ f(1_A) &= 1_B, \end{aligned}$$

where 1_A and 1_B are the units of A and B respectively.

Definition 1.5. *For K-vector spaces V and W the* ***tensorproduct*** *$V \otimes W$ is a vector space with a bilinear map $\phi : V \times W \to V \otimes W$ defined by the following universal property*

$$\begin{tikzcd} V \times W \arrow[rr, "\phi"] \arrow[dr, "f"'] & & V \otimes W \arrow[dl, "\exists!\tilde{f}"] \\ & A & \end{tikzcd}$$

i.e. for every bilinear f from $V \times W$ to a K-vector space A there is exactly one linear $\tilde{f}$ such that $f = \tilde{f} \circ \phi$.

Remark 1.6. *i) The tensorproduct exists and is unique up to isomorphism.*

ii) If $\{v_j\}_{1 \leq i \leq n}$ is a basis of the K-vector space V and $\{w_i\}_{1 \leq i \leq k}$ is a basis of the K-vector space W then $\{v_j \otimes w_i\}_{1 \leq i \leq k, 1 \leq i \leq n}$ is a basis of $V \otimes W$.

Definition 1.7. *Let K be a field and let E be a K-module. For each integer $r \geq 0$ we define:*

$$T^r(E) = \bigotimes_{i=1}^{r} E \text{ and } T^0(E) = K.$$

From the associativity of the tensor product, we obtain a bilinear map:

$$T^r(E) \times T^s(E) \to T^{r+s}(E),$$

which is associative. By means of this bilinear map, we can define an algebra structure on the direct sum

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E).$$

We define $T(E)$ as the ***tensor algebra*** *of E over K.*

Theorem 1.8. *Let A be a unital associative K-algebra and $f : E \to A$ be a linear map, then the tensor algebra has the universal property that there is exactly one algebra homomorpism $\tilde{f} : T(E) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[rr, "i"] \arrow[dr, "f"'] & & T(E) \arrow[dl, "\tilde{f}"] \\ & A & \end{tikzcd}$$

Remark 1.9. *Let V be a vector space of dimension n over K and $\{v_1, ..., v_n\}$ be a basis of V over K. Then the elements*

$$M_i(v) = \underbrace{v_{i_1} \otimes ... \otimes v_{i_r}}_{r-times}, \quad 1 \leq i_v \leq n$$

form a basis of $T^r(V)$ and every element of $T(V)$ has a unique expression as a finite sum

$$\sum_{(i)} a_{(i)} M_{(i)}(v), \quad a_{(i)} \in K$$

with almost all $a_{(i)}$ equal to 0.

Definition 1.10. *Given two algebras A and B we define an algebra structure on the tensor product $A \otimes B$ by*

$$(a \otimes b) \circ (a' \otimes b') = aa' \otimes bb'$$

*this is called **ordinary tensor product** of A and B.*

Lemma 1.11. *The ordinary tensor product actually satisfies the conditions of an algebra.*

Definition 1.12. *A **left-ideal** of a K-algebra is a linear subspace L with the following properties for all $x, y \in L$, $z \in A$ and $c \in K$*

$$\begin{aligned} x + y &\in L \\ cx &\in L \\ z \circ x &\in L. \end{aligned}$$

*With $x \circ z \in L$ we get a **right-ideal**. With both properties L is a **two-sided ideal**.*

Theorem 1.13. *Suppose that A and B are K-algebras and that $f : A \to B$ is an algebra homomorphism. Let I be an ideal such that $I \subset ker(f)$ then there exists a unique algebra homomorphism $\tilde{f}$ such that the following diagramm commutes*

$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[dr, "f"'] & & A/I \arrow[dl, "\tilde{f}"] \\ & B & \end{tikzcd}$$