3d-TQFTs from non-semisimple MTCs

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Structure: 1) prelims 2) Modify Lyubashenko's invariant
3) Extend to TFT

$\Bbbk$ - alg closed field

1. Preliminaries

1.1 FTCs + graphical calculus

Recall: A finite tensor category is a rigid monoidal abelian $\Bbbk$-linear cat s.t.

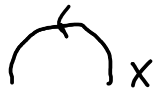
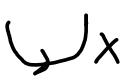
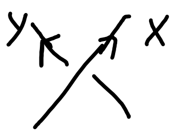
- $\dim \mathcal{C}(X,Y) < \infty$, Length $(X) < \infty$ $\forall X, Y \in \mathcal{C}$
- $\otimes$ is bilinear
- finite set Irr of repr. of iso classes of simple objects, and $\mathbb{1} \in$ Irr
- every $U \in$ Irr has a projective cover $(P_U, p_U : P_U \twoheadrightarrow U)$

Think: $\mathcal{C}$ = H-mod H Hopf algebra

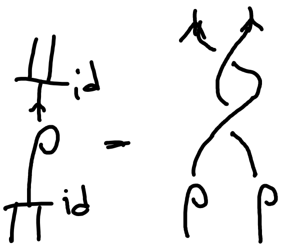
For the rest of this section: $\mathcal{C}$ ribbon FTC s.t. strictly pivotal.

$\rightsquigarrow$ two sided dual $X^\vee$ of $X \in \mathcal{C}$

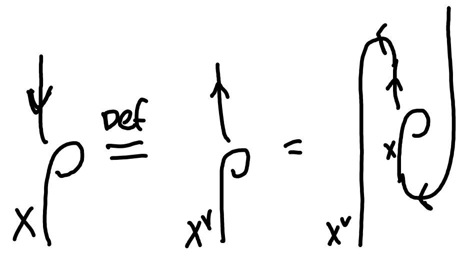
Our graphical notation is (read: bottom to top)

- left eval/coeval for X :  , 
- right eval/coeval for X :  , 
- braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$: 
- twist $\vartheta_X : X \to X$: 

so that eg the twist axioms are



and



1.2 Tensor ideals and traces

A full subcategory $I \subseteq \mathcal{C}$ is called an <u>ideal</u> if

1) Closure under retracts: $A \oplus B \in I \Rightarrow A, B \in I$

2) $A \in I,\ X \in \mathcal{C} \Longrightarrow A \otimes X \in I$ (i.e. $I \otimes \mathcal{C} \subseteq I$)

A <u>trace t on an ideal $I \triangleleft \mathcal{C}$</u> is a family of linear maps

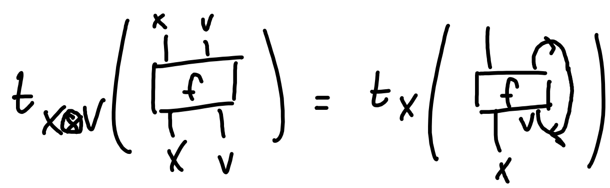
$$\{ t_X : \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \mathbb{k} \}_{X \in I}$$

satisfying

1) Cyclicity: $\forall X, Y \in I$, $f, g^{op} \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,X)$: $t_X(g \circ f) = t_Y(f \circ g)$

2) right partial trace: $X \in I,\ V \in \mathcal{C}$.

$\forall f \in \mathrm{End}_{\mathcal{C}}(X \otimes V)$

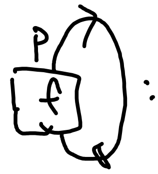


A trace t on I is called <u>non-degenerate</u> if the pairing

$$t_V(- \circ -) : \mathcal{C}(W,V) \times \mathcal{C}(V,W) \longrightarrow \mathbb{k}$$

is non-degenerate $\forall V \in I,\ W \in \mathcal{C}$.

<u>Example</u> 1) $I = \mathcal{C}$, $t = tr$. Then $\forall$ projective P, $f \in \mathrm{End}_{\mathcal{C}}(P)$:

$tr_P(f) =$  $: \mathbb{1} \longrightarrow P \otimes P^\vee \twoheadrightarrow \mathbb{1}$

$\Longrightarrow$ tr non-degen $\Rightarrow \mathcal{C}$ semisimple (ss.)

2) Let $\mathcal{C}$ be unimodular $(P_{\mathbb{1}}^\vee \cong P_{\mathbb{1}})$. $I = \mathrm{Proj}(\mathcal{C})$

<u>Thm</u> (Geer + al):

$\exists!$ (up to $\mathbb{k}^\times$) non-zero trace t on I. It is non-degenerate.

⌜ $\mathcal{C} = H\text{-mod} \rightsquigarrow \mathrm{Proj}(\mathcal{C}) = $ proj. H-modules, t is completely determined by cointegral $\lambda \in H^*$ ⌟

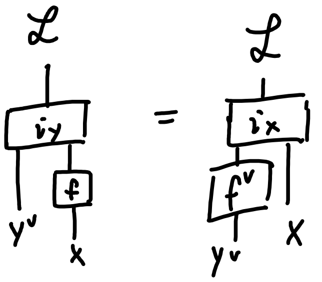
1.3 Modular Tensor categories

In $\mathcal{C}$, the coend and end

$$(\mathcal{L}, i) = \int^{X \in \mathcal{C}} X^\vee \otimes X \,, \qquad (\mathcal{E}, j) = \int_{X \in \mathcal{C}} X \otimes X^\vee$$

exist.

⌜Recall that e.g.

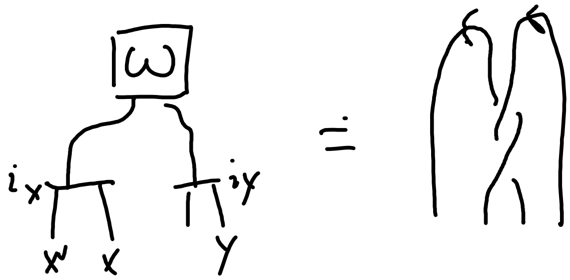


and $\mathcal{L}$ is universal wrt this property:

$$\mathrm{Dinat}(-^\vee \otimes -, V) \cong \mathcal{C}(\mathcal{L}, V)$$ ⌟

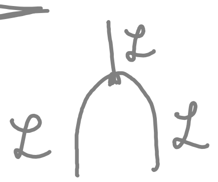
For the rest of the talk: assume $\mathcal{C}$ is modular,

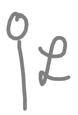
i.e. the pairing $\omega : \mathcal{L} \otimes \mathcal{L} \to \mathbb{1}$,



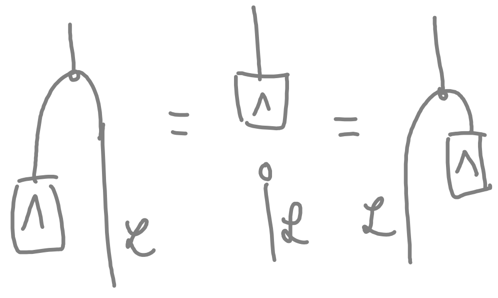
, induces an iso $\mathcal{L} \cong \mathcal{L}^\vee$

$\rightsquigarrow \exists$ special morphism $\Lambda : \mathbb{1} \to \mathcal{L}$, called the <u>integral</u>





$\Longrightarrow$



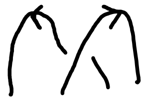
2. Modifying Lyubashenko's invariant

manifolds are oriented, diffeomorphisms are positive, links/tangles are oriented + framed

2.1 Bichrome ribbon graphs and the Lyubashenko-Reshetikhin-Turaev functor

Def (i) An <u>n-bottom tangle</u> is something like this ($n=2$)

✓, not

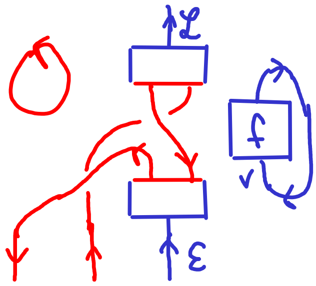


✗

(ii) An <u>n-bottom graph</u>: ribbon graph w/

Edges = {red} ∪ {blue}, Coupons = {bichrome} ∪ {blue}

+ conditions (e.g. $n=1$)

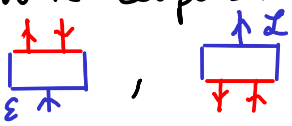


(a) The $2n$ leftmost incoming legs are red, all other boundary legs are blue. ✓

(b) Red edges unlabeled, blue labeled by ob $\mathcal{C}$ ✓

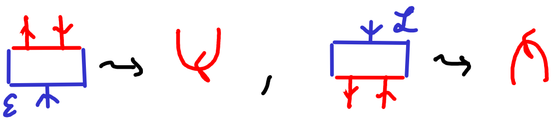
(c) Blue coupons: labeled by mor $\mathcal{C}$,

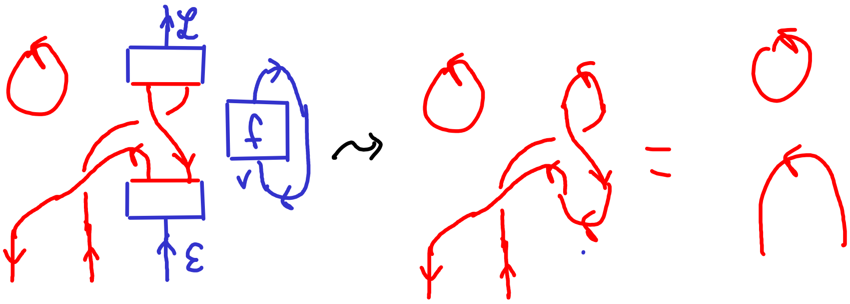
bichrome coupons: unlabeled, two types



(d) "Smoothing" yields an n-bottom tangle:

- throw away purely blue stuff
- replace



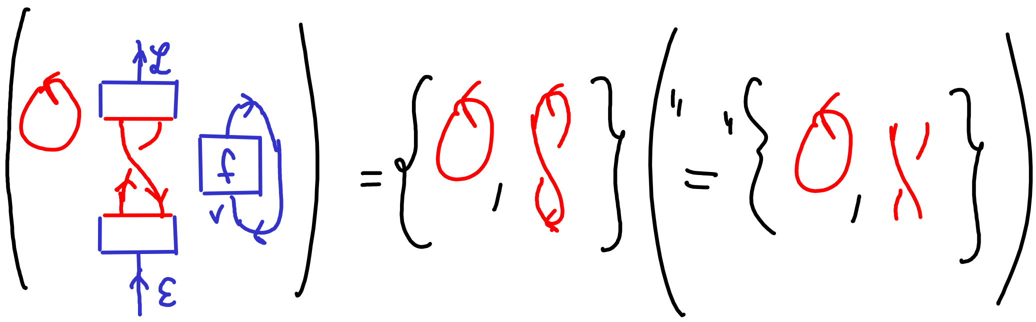


(iii) bichrome graph := 0-bottom graph

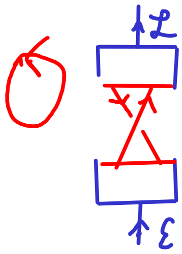
How to interpret a bichrome graph as a morphism in $\mathcal{C}$

First: Cycle of bichr. graph T $\hat{=}$ a connected component in smooth(T)

eg: cycles

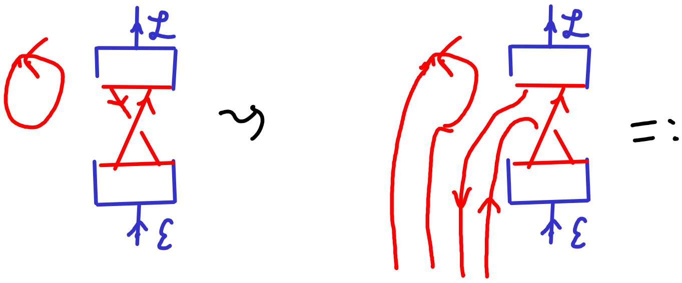


Consider $T =$



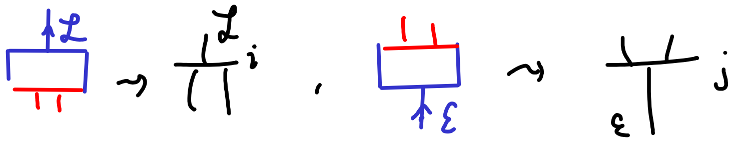
$(\rightsquigarrow \mathcal{C}(\varepsilon, \mathcal{L}))$

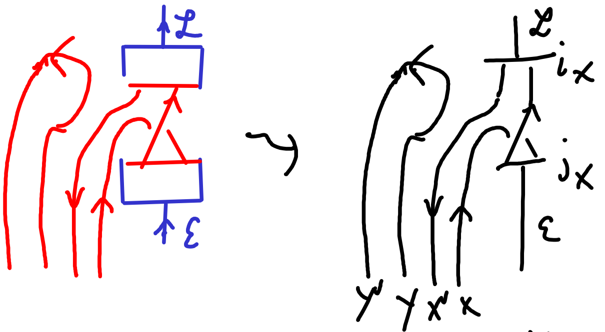
Step 1 For each cycle c: cut an edge of c open, and bend down



$=: \tilde{T}$, a 2-bottomgraph

Step 2 Forget color,





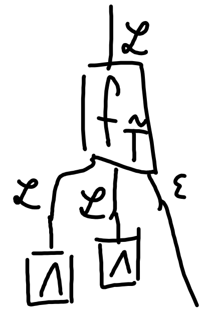
$\in \mathrm{Dinat}\left[(-^\vee \otimes -)\otimes(-^\vee \otimes -)\otimes \varepsilon,\ \mathcal{L}\right]$

$=: \eta^{\tilde{T}}$

Step 3 $\mathrm{Dinat}\left[(-^\vee \otimes -)\otimes(-^\vee \otimes -)\otimes \varepsilon,\ \mathcal{L}\right] \cong \mathcal{C}(\mathcal{L}\otimes\mathcal{L}\otimes\varepsilon, \mathcal{L})$

$\eta^{\tilde{T}} \longmapsto f_{\tilde{T}}$

Step 4

$F_\Lambda(T) :=$  $\in \mathcal{C}(\varepsilon, \mathcal{L})$

Prop $F_\Lambda(T)$ is well-defined.

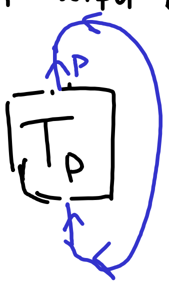
3.2 Renormalized Lyubashenko invariant of closed 3-manifolds

Fix $I = \mathrm{Proj}(\mathcal{C}) \trianglelefteq \mathcal{C}$, non-degen trace t on I.

A closed bichrome graph is <u>admissible</u> if ≥ 1 blue edge colored by $P \in \mathrm{Proj}(\mathcal{C})$.

(Rem: $\mathcal{C}$ ssi $\Rightarrow$ admissible iff nonempty)

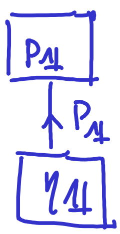
Thm T adm bichr. graph with edge colored by projective P.

e.g. $T =$ 

Then $F'_{\Lambda,t}(T) = t_P\left(F_\Lambda\left(\boxed{T_P}\right)\right)$ is an isotopy inv of T. //

The difference between F_Λ and $F'_{\Lambda,t}$

Fact $\mathcal{C}$ modular $\Rightarrow \exists\, \eta_{\mathbb{1}} : \mathbb{1} \to P_{\mathbb{1}}$ non-zero (recall: $p_{\mathbb{1}} : P_{\mathbb{1}} \twoheadrightarrow \mathbb{1}$)

Consider the bichr. graph $T =$  closed, admissible

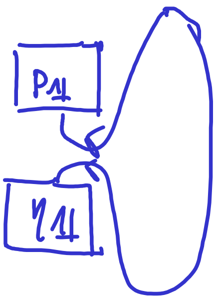
- $F_\Lambda(T) = \mathbb{1} \xrightarrow{\eta_{\mathbb{1}}} P_{\mathbb{1}} \xrightarrow{p_{\mathbb{1}}} \mathbb{1} = 0$ unless $\mathcal{C}$ ssi

- $$F'_{\Lambda,t}(T) = t_{P_{\mathbb{1}}}\left(\begin{array}{c} \uparrow P_{\mathbb{1}} \\ \boxed{\eta_{\mathbb{1}}} \\ \boxed{p_{\mathbb{1}}} \\ \uparrow \end{array} \right) \neq 0$$

because:
- $\mathcal{C}(\mathbb{1}, P_{\mathbb{1}}) = \Bbbk\, \eta_{\mathbb{1}}$, $\mathcal{C}(P_{\mathbb{1}}, \mathbb{1}) = \Bbbk\, p_{\mathbb{1}}$
- t non-degenerate

Caveat:

$$F'_{\Lambda,t}(T \otimes T) = t_{P_{\mathbb{1}}}\left(\begin{array}{c} \uparrow P_{\mathbb{1}} \\ \boxed{\eta_{\mathbb{1}}} \\ \boxed{p_{\mathbb{1}}} \\ \uparrow \end{array} \cdot F_\Lambda(T) \right) = 0$$



<u>Thm</u>
- M – closed connected 3-mfld obtained by surgery along a red l-component link L of signature $\sigma(L)$
- T – admissible closed bichrome graph in M

Then
$$L'_{\mathcal{C},I}(M,T) := \mathcal{D}^{-1-l}\, \delta^{-\sigma(L)}\, F'_{\Lambda,t}(L \cup T)$$

is a topological invariant of (M,T). ∎

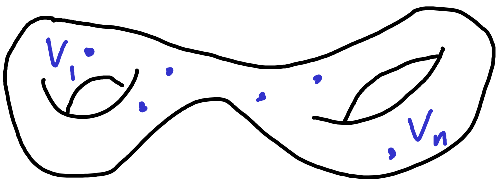
<u>Rem</u>: $\mathcal{C}$ ssi $\Rightarrow$ $L'_{\mathcal{C},I} = L^{RT}_{\mathcal{C}}$

3. Extending the invariant to a (2+1)-TFT

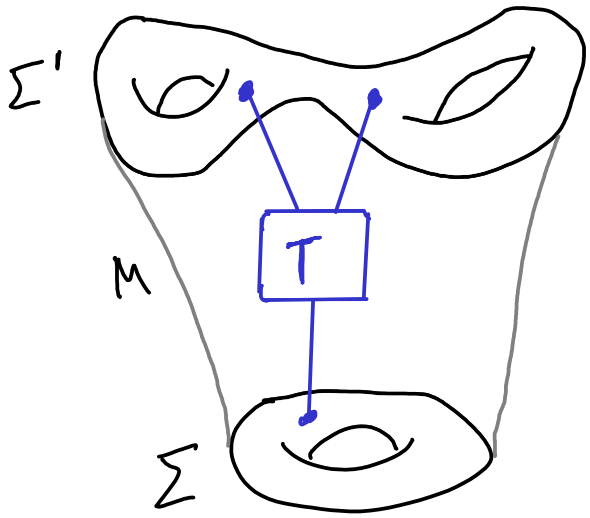
Here, $\mathcal{C}$ is modular, and we fix $I = \mathrm{Proj}(\mathcal{C})$ w/ modified trace t and the normalization $t_{P_{\mathbb{1}}}(\eta_{\mathbb{1}} \circ \varepsilon_{\mathbb{1}}) = 1$.

The cobordism category $\mathrm{Cob}_{\mathcal{C}}$ has

- objects: $\boldsymbol{\Sigma} = (\Sigma, P_{\mathcal{C}}\ \mathrm{ob}\,\mathcal{C}, \cancel{L})$



- morphisms: $\mathbb{M}: \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}'$ eq. class of (M, T, n)



T bichrome

- $\otimes = \sqcup$

$\rightsquigarrow$ Symmetric, rigid

$\mathbb{M} \in \mathrm{mor}\,\mathrm{Cob}_{\mathcal{C}}$ is <u>admissible</u> if every connected component of $\mathbb{M}$ disjoint from incoming boundary contains an admissible graph.

$\rightsquigarrow$ We get the <u>admissible cobordism category $\check{\mathrm{Cob}}_{\mathcal{C}}$</u> by restricting $\mathrm{Cob}_{\mathcal{C}}$ to admissible cobordisms.

For $\mathbb{M} = (M, T, n) \in \check{\mathrm{Cob}}_{\mathcal{C}}$ w/ M closed can define

$$L'_{\mathcal{C}}(\mathbb{M}) := \delta^n L'_{\mathcal{C}}(M, T)$$

$\rightsquigarrow$ Universal construction of BHMV yields functor

$$V_{\mathcal{C}} : \check{\mathrm{Cob}}_{\mathcal{C}} \longrightarrow \mathrm{Vect}_{\mathbb{k}}$$

$\ulcorner$ it maps Σ to a certain quotient of $\mathbb{k}\check{\mathrm{Cob}}_{\mathcal{C}}(\emptyset, \Sigma)$ $\lrcorner$

<u>"Thm"</u> $V_{\mathcal{C}}$ is sym. monoidal. It maps $\Sigma_{g,n}$ decorated with $V_1, \dots, V_n \in \mathcal{C}$ to a vector space iso to

$$\mathcal{C}(\mathcal{L}^{\otimes g} \otimes V_1 \otimes \dots \otimes V_n, \mathbb{1})^* \quad //$$

$\Sigma_{g,n}$ as above $\rightsquigarrow$ get an L s.t.

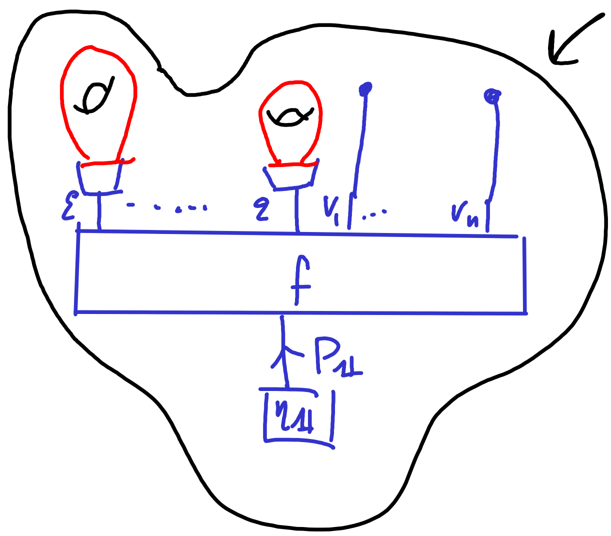
$$\mathbf{\Sigma}_{g,n} = (\Sigma_{g,n}, \{(+, V_1), \dots, (+, V_n)\}, L)$$

$$\begin{tikzcd}
 & \mathbb{k}\check{\mathrm{Cob}}_{\mathcal{C}}(\emptyset, \mathbf{\Sigma}_{g,n}) \arrow[dr, two heads] & \\
\mathcal{C}(P_{\mathbb{1}}, \varepsilon^{\otimes g} \otimes V_1 \otimes \dots \otimes V_n) \arrow[ur, "\xi"] \arrow[rr, dashed, two heads, "\exists"'] & & V_{\mathcal{C}}(\mathbf{\Sigma}_{g,n})
\end{tikzcd}$$

from univ. construction

$$f : P_{\mathbb{1}} \longrightarrow \mathcal{E}^{\otimes s} \otimes V_1 \otimes \cdots \otimes V_n$$

∂handlebody $= \Sigma_S$

$\xi(f) =$ 

$: \emptyset \longrightarrow \Sigma_{g,n}$