Exercise sheet # 12
Algebraic Geometry SS 2018

(Ingo Runkel)

Exercise 49 (3 P)

Let X, Y be varieties and $p \in X$, $q \in Y$. Suppose the local rings $\mathcal{O}_{p,X}$ and $\mathcal{O}_{q,Y}$ are isomorphic as k-algebras. Show that X and Y are birational.

Hint: Reduce the question to the case of affine varieties.

Exercise 50 (7 P)

Let $n \geq 2$ and $M = Z(\{X^2 - Y^n\}) \subset \mathbb{A}^2$. Is M irreducible? If not, what are its irreducible components? Is it (or its irreducible components) isomorphic to $\mathbb{A}^1$? Birational to $\mathbb{A}^1$? For irreducible M let $\tilde{M}$ be the blow-up of M at 0. Give an affine variety in $\mathbb{A}^2$ isomorphic to $\tilde{M}$. Is $\tilde{M}$ isomorphic to $\mathbb{A}^1$?

Exercise 51 (4 P)

Compute the blow up of the cone $M = Z(\{X^2 - YZ\}) \subset \mathbb{A}^3$ at zero. What is its intersection with $\pi^{-1}(\{0\})$? (Here, $\pi : \tilde{\mathbb{A}}^3 \to \mathbb{A}^3$ is the projection from the blow-up of $\mathbb{A}^3$ to $\mathbb{A}^3$.) Is there a general statement about $\pi^{-1}(\{0\})$ for blow-ups of cones at 0 (possibly defined in terms of more than one homogeneous polynomial)?

Exercise 52 (0 P)

Prove the Cayley-Hamilton Theorem using methods from algebraic geometry.

Please turn over.

Exercise 53 (10 P)

Let L/K be a field extension, i.e. L, K are fields and $K \subset L$. A *transcendence basis* of L over K is a subset $S \subset L$ such that S is algebraically independent over K and L is finite over $K(S)$ (cf. Section 1.4). Here $K(S)$ denotes the subfield of L generated by K and S.
We say that an element $a \in L$ is algebraically dependent on S (over K) if a is the zero of a non-zero polynomial with coefficients in $K(S)$. Equivalently (why?) $(K(S))[a]$ is finite over $K(S)$.

Suppose L is finitely generated over K. Show:

1. Let K be infinite. L has a finite transcendence basis over K.

   Hint: While L is finitely generated over K as a field, it is not necessarily finitely generated as an algebra. (Why are these notions different?). But maybe one can still reduce the question to Noether normalisation.

2. Every transcendence basis of L over K is finite.

3. Any two finite such transcendence bases have the same number of elements.

Hint: For parts 2 and 3 one can proceed as follows.

(a) Show the following exchange lemma: Let $\{a_1, \dots, a_n\} \subset L$ be a subset and let $b \in L$. Suppose that b is algebraically dependent on $\{a_1, \dots, a_n\}$ but not on $\{a_1, \dots, a_{n-1}\}$. Then a_n is algebraically dependent on $\{a_1, \dots, a_{n-1}, b\}$. (To do so, consider a polynomial relation $f(a_1, \dots, a_n, b) = 0$ as a polynomial in the last two entries.)

(b) Show: Let $S \subset L$ be an algebraically independent set over K. Suppose a is algebraically dependent on S and b is algebraically dependent on $S \cup \{a\}$. Then b is algebraically dependent on S. (Why is $K(S)[a, b]$ finite over $K(S)$?)

(c) Consider two transcendence bases X and Y, and suppose $X = \{a_1, \dots, a_n\}$ is finite and smaller than Y. Carry out a recursive argument: If Y contains the elements $\{a_1, \dots, a_l\}$ of X, then we can build Y' with the same number of elements which now contains $\{a_1, \dots, a_{l+1}\}$.